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CONTENTS

ON POPULAR METHODS AND EXTANT PROBLEMS IN THE SOLUTION OF POLYNOMIAL EQUATIONS

Donald Greenspan

1. *Introduction.* Problems of stability [30, p. 126] and eigenvalues [17, p. 64], and the satisfaction of esthetic tastes still have mathematicians concerned with the question of completely solving the polynomial equation:

$$P(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n = 0, \tag{1.1}$$

where the a_i, $i = 0, 1, \cdots, n$ may be complex and $a_n \neq 0$. At times one will wish to write (1.1) in decreasing powers of z and this will be denoted by

$$\downarrow P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0. \tag{1.2}$$

The preliminary attempt to solve (1.1) usually involves a host of well known algebraic theorems which include: Remainder Theorem, Factor Theorem, Descartes' Rule of Signs, Sturm's Theorem, and various theorems on possible rational roots and the occurence in pairs of real or complex roots. Nevertheless, examination of such simple equations as:

$$z^8 - 3iz^6 - \sqrt{2}\, z^3 + (1+i)z^2 - (1-\sqrt{3})z + 5 = 0, \text{ and}$$

$$10157z^{15} - 342.37z^{14} + 899.67z^7 - 6754.7z^6 + 5007.9z^3 + 3482.1z + 8103.5 = 0$$

demonstrates immediately the inefficacy of these theorems in finding all the roots.

However, at present, with the advent of modern computational devices, new procedures for solving (1.1) have emerged and older ones, once considered folly, have entered into the realm of practicality. Let us examine these, and other interesting techniques, and, where possible, indicate vital remaining problems and inadequacies. The only two assumptions made are that $a_n = 1$ and that the reader who is interested in detailed examples will consult the references given with each technique, where abundant examples exist.

2. *Technique of Dividing Out Quadratic Factors.* One might attempt to factor $P(z)$ or $\downarrow P(z)$ into linear [1], [46], quadratic [1], [18], [22], [25], [28], [29], [31], [37], [45], cubic or higher order factors [1], and then to find the zeros of these factors. A popular attempt at present is to try to

find quadratic factors of $P(z)$ or $\downarrow P(z)$. Three such procedures follow and a graphical procedure has been placed in Section 9.

(2a) Bairstow's Method: Guess a trial quadratic factor: z^2-pz-t. Let

$$P(z) = (z^2-pz-t)q(z)+q_1(z)+(q_0-pq_1);$$

$$q(z) = (z^2-pz-t)T(z)+T_1z+(T_0-pT_1).$$

To determine a new quadratic factor: $z^2-(p+\delta p)z-(t+\delta t)$, use the formulas:

$$D = T_0^2-MT_1 \quad , \qquad D\delta t = Mq_1-T_0q_0 ,$$

$$M = tT_1+pT_0 \quad , \qquad D\delta p = T_1q_0-T_0q_1 .$$

The process then repeats in the same fashion using the new quadratic factor and terminates when a quadratic factor yields $\delta p = \delta t = 0$. This quadratic factor is then factored out of $P(z)$ and readily yields, therefore, two roots and an equation degree $(n-2)$ to be solved, or else, one may try to find other quadratic factors of the original equation.

(2b) Lin's Method: Guess a trial quadratic factor: z^2+pz+t. Divide $\downarrow P(z)$ by this factor but carry the division only down to the point where the remainder is a quadratic expression. (The usual division process of obtaining a linear or constant remainder is thus avoided.) Divide the quadratic expression by the coefficient of its z^2 term. Using this result, repeat the process until a trial factor yields itself for a new trial factor.

(2c) Friedman's Method: Guess a trial quadratic factor: z^2+pz+t. Divide $\downarrow P(z)$ by the quadratic, yielding:

$$\downarrow P(z) = Q(z)(z^2+pz+t) + Rz + S.$$

Now write $Q(z)$ in increasing powers of z and divide it into $P(z)$, yielding:

$$P(z) = (t_1+p_1z+q_1z^2)Q(z) + R(z).$$

Reduce the factor $(t_1+p_1z+q_1z^2)$ by dividing by q_1 to give: $z^2+p_2z+t_2$. Use: $z^2+p_2z+t_2$, as a new trial factor and iterate the process.

The primary question is, of course, one of convergence. The question: "How should I make the first guess to be sure of convergence?" is unanswered. There are papers like [33] which set forth conditions for convergence, but all claims start with a key phrase which assumes that the original guess was made such that if p and t are in error ϵ, then ϵ^2 and higher order terms can be neglected. Unfortunately such a statement does not clarify the problem of how to choose the first p and t.

3. *General Iteration Formula*: $z_i = G(z_{i-1})$. Given (1.1), solve it for z:

$$z = -\frac{a_0}{a_1}-\frac{a_2}{a_1}z^2-\frac{a_3}{a_1}z^3 \cdots -\frac{a_n}{a_1}z^n. \tag{3.1}$$

Consider then:

$$z_i = -\frac{a_0}{a_1} - \frac{a_2}{a_1} z_{i-1}^2 - \frac{a_3}{a_1} z_{i-1}^3 - \cdots - \frac{a_n}{a_1} z_{i-1}^n . \tag{3.2}$$

(If $a_1 = 0$, use:

$$z_i = a_0 + z_{i-1} + a_2 z_{i-1}^2 + a_3 z_{i-1}^3 + \cdots + a_n z_{i-1}^n .) \tag{3.3}$$

Make an initial guess z_0 and use (3.2), (or (3.3), as the case may be), to iterate. Stop if and when a value z_q yields z_{q+1} such that $z_q = z_{q+1}$.

This method again is one for which there exists no "adequate" sufficient conditions for convergence, even though, like Lin's method, it has been found to converge often enough in practice to be considered an important technique. Sufficient conditions, though, do exist, as, for example, those of Goodstein and Broadbent [20], which are stated for more general functions than polynomials.

It is interesting to note that once a value is chosen and convergence is assured, the convergence may be accelerated by means of various formulas as that of Samuelson [43]:

In place of $x_{t+1} = G(x_t)$, use:

$$x_{t+1} = \frac{\{G[G(x_t)]\}^2 - G(x_t)G\{G[G(x_t)]\}}{2G[G(x_t)] - G(x_t) - G\{G[G(x_t)]\}} = F(x_t)$$

4. *Rule of False Position.* Though this technique is strictly analytic, a graphical description will be more revealing. Suppose a_i are real in (1.1). Graph $y = P(x)$. Suppose by some means x_0 and x_1 have been found such that $P(x_0) > 0$, $P(x_1) < 0$. (See diagram.) Let $P(x_0) = y_0$, $P(x_1) = y_1$. Draw the chord joining the two points (x_0, y_0), (x_1, y_1). Let this chord intercept the X-axis in the point x_2. If $P(x_2) = 0$ we are done; if $P(x_2) > 0$, iterate the process using x_1 and x_2; if $P(x_2) < 0$, iterate using x_0 and x_2. Continue in this way until a real root is obtained to any desired degree of accuracy.

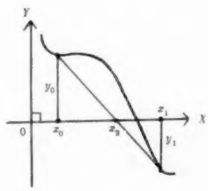

Diagram 1.

Variations of this technique exist [27], but all are of limited use. However, a generalization called the Downhill Method [61], which yields both real and complex roots, offers much promise.

5. *Aitken's Method* [2], [37]. Suppose in (1.1) the a_i are real. Aitken's basic theorem is:

$$\lim_{t\to\infty} \frac{\begin{vmatrix} P(t+1) & P(t+2) \cdots\cdots & P(t+m) \\ P(t) & P(t+1) \cdots\cdots & P(t+m-1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ P(t-m+2) & \cdots\cdots\cdots\cdots & P(t+1) \end{vmatrix}}{\begin{vmatrix} P(t) & P(t+1) \cdots\cdots & P(t+m-1) \\ P(t-1) & P(t) \cdots\cdots & P(t+m-2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ P(t-m+1) & \cdots\cdots\cdots\cdots & P(t) \end{vmatrix}} = z_1 \cdot z_2 \cdots\cdots z_m,$$

where $z_1, z_2, \cdots, z_m$ are the first m of the roots of (1.1) arranged in non-ascending order of absolute value.

Bernoulli's procedure is the special case $m = 1$.

Aitken's technique is to proceed as follows: If $|z_1| > |z_2|$, then we calculate $f(t)$, $f(t+1)$, $f(t+2)$, $\cdots$, and as $t\to\infty$, $\frac{f(t+1)}{f(t)} \to z_1$.

Compute
$$f_2(t) = \begin{vmatrix} f(t) & f(t+1) \\ f(t-1) & f(t) \end{vmatrix}, \cdots,$$

$$f_{s+1}(t) = \begin{vmatrix} f_s(t) & f_s(t+1) \\ f_s(t-1) & f_s(t) \end{vmatrix} \div f_{s-1}(t), \quad s \geqq 2.$$

Let $E_s(t) = \frac{f_s(t+1)}{f_s(t)}$, $|z_s| > |z_{s+1}|$, then $\lim_{t\to\infty} E_s(t) = z_1 \cdot z_2 \cdots z_s$. So we determine

$$\begin{array}{l} E_1(t) \sim z_1 \\ E_2(t) \sim z_1 z_2 \\ E_3(t) \sim z_1 z_2 z_3 \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \end{array}$$

, and then deduce the roots.

The presence of complex roots is indicated by fluctuations in sign and

magnitude in the values of f_m and $E_m(t)$. Consider the simple case where the roots are z_1, z_2, $re^{i\theta}$, $re^{-i\theta}$, z_5, $\cdots$, and $|z_1| > |z_2| > |z_3| = |z_4| > |z_5|$. Then $E_1(t) \to z_1$, $E_2(t) \to z_1 z_2$, $E_3(t)$ fluctuates, $E_4(t) \to z_1 z_2 z_3 z_4$. Hence $z_1 z_2 r^2 = z_1 z_2 z_3 z_4$ and r is determined immediately. θ can then be found as follows:

$$\frac{c(t+1)\,c(t)+1}{2\,c(t)} \to \cos\theta \quad \text{, where} \quad c(t) = \frac{E_3(t)}{z_1 z_2 r}.$$

The cases of real multiple roots, multiple pairs of complex roots, and roots of the same modulus but different amplitudes are not easily recognizable, and the case of many roots with the same amplitude, some with the same modulus, others with different moduli, easily escapes all detection.

6. *Eigenvalue Method.* Not only does the problem of matrix eigenvalues lead to the question of solving a polynomial equation, but the problem of solving (1.1) could be answered if the eigenvalues of a matrix could be evaluated directly. One can easily show that (1.2) is the characteristic equation of the matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots\cdots\cdots\cdots & & & \\ 0 & 0 & 1 & 0 & \cdots\cdots\cdots\cdots & & & \\ 0 & 0 & 0 & 1 & \cdots\cdots\cdots\cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \cdots\cdots\cdots\cdots & & & \\ 0 & 0 & 0 & 0 & \cdots\cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdots\cdots & 0 & 1 & 0 \\ -a_0 & -a_1 & -a_2 & \cdot & \cdots\cdots & \cdot & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

It is necessary, of course, as was assumed in section (1), that $a_n = 1$. If it is known that all the roots are real, and of different absolute value, then the following direct method may be used [17]. Begin with a vector $x_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. Calculate: Ax_1 to give a vector y_1. Write $y_1 = k_1 x_2$, where x_2 has 1 for its last component. Calculate $Ax_2 = y_2$. Write $y_2 = k_2 x_3$, where

x_3 has 1 for its last component. Continue the process until k_i converges to an eigenvalue of the matrix and hence a root of (1.2).

7. *Newton's Method*. Of all approaches, Newton's method occupies a most endeared position. The usual textbook discussion consists of drawing a misleading diagram in which a guess x_{n-1} leads to drawing a tangent to the curve $y = \dotplus P(x)$ at $(x_{n-1}, \dotplus P(x_{n-1}))$, and this tangent then intersects the X-axis at a point x_n which is a better estimate to the root x. The analytic statement of this geometric process is given by:

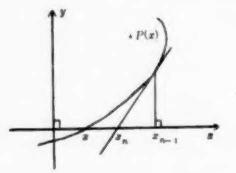

Diagram 2

$$(7.1)\quad x_n = x_{n-1} - \frac{\dotplus P(x_{n-1})}{\dotplus P'(x_{n-1})} \equiv x_{n-1} - \frac{P(x_{n-1})}{P'(x_{n-1})}.$$

Again we see that for the process to work, x_0 must be chosen properly, and herein lies the problem. For example, consider the quadratic equation with real a_i, and roots $\pm ci$, $c \neq 0$ and real. Then any initial guess x_0 which is real can never converge to the roots, for Newton's formula will then yield only real numbers. Hence an example can easily be set up such that the roots $\pm ci$ are arbitrarily close to the initial x_0, yet no convergence would occur. The following sufficient conditions have thus far been established:

(7a) Theorem of Mysovkeh [34]: Let $P(x)$ be twice differentiable and $P(x_0)P'(x_0) > 0\{P(x_0)P'(x_0) < 0\}$; $P''(x)$ exists in $I(x_0 - \lambda, x_0)\{I(x_0, x_0+\lambda)\}$; $|P''(x)| \leqq k$; $|\frac{1}{P'(x)}| \leqq P$ on I; $|P(x_0)| < \eta$; and $h = B^2K\eta \leqq 4$. Then $P(x) = 0$ has a unique solution defined by

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)},\quad n = 0, 1, \cdots, \text{ and it converges.}$$

The constant 4 in $h \leqq 4$ cannot be improved.

(7b) Theorem of Ostrowski (unpublished American University notes used as working paper at National Bureau of Standards, Washington, D.C., 1952).

$$\text{Let } P(x_0)P'(x_0) \neq 0,\ h_0 = -\frac{P(x_0)}{P'(x_0)},\ J_0 : \langle x_0,\ x_0 + 2h_0 \rangle,\ x_1 = x_0 + h_0;$$

P'' exists in J_0, $\underset{x \epsilon J_0}{\text{Max}} |P''(x)| = M$; $\underset{x \epsilon J_0}{|P''(x)|} \leqq |\frac{P'(x_0)}{2h_0}|$, i.e., $2|h_0|M \leqq |P'(x_0)|$, then:

$$x_{v+1} = x_v - \frac{P(x_v)}{P'(x_v)} \to \xi, \quad v = 0, 1, 2, \cdots,$$

where ξ is the only root in J_0; ξ will be simple unless it lies on the boundary.

(7c) Generalization of (7b) by Ostrowski (same reference) : Let $P(z_0)P'(z_0) \neq 0$, $h = -\dfrac{P(z_0)}{P'(z_0)}$, $z_1 = z_0 + h_0$; $K_0 : |z - z_1| \leq h_0$; $P(z)$ analytic in K_0, $\underset{z \epsilon K_0}{\text{Max}} |P''(z)| = M$; $\underset{z \epsilon K_0}{|P''(z)|} \leq |\dfrac{P'(z_0)}{2h_0}|$, i.e., $2|h_0| M \leq |P'(z_0)|$, then $z_{v+1} = z_v - \dfrac{P(z_v)}{P'(z_v)} \to \epsilon$, $v = 0, 1, \cdots$, where ϵ is the only root in K_0 and is simple unless it lies on the boundary of K_0.

(7d) Theorem of Rosenbloom [39]. If $|P(z_1)| < a$, then: $z_{v+1} = z_v - \dfrac{P(z_v)}{P'(z_v)}$, converges to a root of (1.1), where :

$$a = \min\left(\frac{1}{2nCE^n}, \frac{1}{4nDE^n}, 1, \frac{\lambda^2}{g}\right),$$

$$\lambda = \frac{1}{2nDE^n}, \quad g = 2\gamma AE^n,$$

and it is assumed that $P(z)$ and $P'(z)$ are relatively prime, that $C(z)P(z) + D(z)P'(z) \equiv 1$, where degrees of $C(z)$ and $D(z)$ are at most n, and if

$$C(z) = \sum_{k=0}^{n} c_k z^k, \quad D(z) = \sum_{k=0}^{n} d_k z^k,$$

that $C = \max |c_k|$, $D = \max |d_k|$, $A = \max |a_k|$, $E = nA + 1$, $n \geq 2$, Moreover, $\gamma =$ binomial coefficient $C_{n+1,q}$, with $q = [(N+1)/2]$.

Other work on convergence intervals has been done by Gorn [21] and Obrechkoff [36].

To extract roots of multiplicity p, Newton's formula is modified to:

$$x_{v+1} = x_v - p\frac{P(x_v)}{P'(x_v)}.$$

8. *Graeffe's Method.* One of the older, more entrenched methods is that of Graeffe. Bodewig [9a] gives a lucid discussion of the various, and there

may be many, facets and complications in the use of this technique. Consider equation (1.2). Construct a new equation which has for roots the squares of the roots of (1.2). This is done readily as follows: Multiply $\downarrow P(z)$ by $(-1)^n \downarrow P(-z)$. A neat way of carrying through this multiplication is shown in [44]. Now let us see what results:

$$\downarrow P(z) = (z - z_1)(z - z_2) \cdots (z - z_n);$$

z_i, $i = 1, 2, \cdots, n$, being the roots of (1.2),

$$(-1)^n \downarrow P(-z) = (-1)^n(-z - z_1)(-z - z_2) \cdots (-z - z_n)$$

$$= (z + z_1)(z + z_2) \cdots (z + z_n)$$

$$(-1)^n \downarrow P(-z) \downarrow P(z) = (z^2 - z_1^2)(z^2 - z_2^2) \cdots (z^2 - z_n^2) = \downarrow P_1^*(y),$$

where $y = z^2$ and $\downarrow P_1^*$ is of degree n in y. The roots of $\downarrow P_1^*(y)$ are the squares of the roots of (1.2). The process is then repeated and yields equations whose roots are the roots of the original equation raised to a power of 2. In practice one does the following. Suppose after n steps we have an equation:

$$v^n + c_{n-1}v^{n-1} + c_{n-2}v^{n-2} + \cdots + c_1v^1 + c_0 = 0. \tag{8.1}$$

On applying the root squaring technique we secure an equation:

$$w^n + d_{n-1}w^{n-1} + d_{n-2}w^{n-2} + \cdots + d_1w^1 + d_0 = 0. \tag{8.2}$$

If $d_j = c_j^2$, for $j = 0, 1, \cdots, n-1$, then the process has worked completely and we find the roots from the equations:

$$\begin{aligned} v + c_{n-1} &= 0, \\ c_{n-1}v + c_{n-2} &= 0, \\ c_{n-2}v + c_{n-3} &= 0, \\ &\;\;\vdots \\ c_1v + c_0 &= 0. \end{aligned}$$

Once these v's are calculated, the original z_i's can be found since the v's are a known power of the z_i's.

Now suppose the $d_j \neq (c_j)^2$ for all $j = 0, 1, \cdots, n-1$. Then the procedure is as follows: Consider (8.2) If $d_{n-1} = c_{n-1}{}^2$, then write down:

$$v + c_{n-1} = 0$$

If $d_{n-1} \neq c_{n-1}{}^2$, $d_{n-2} = c_{n-2}{}^2$, write down:

$$v^2 + c_{n-1}v + c_{n-2} = 0$$

If $d_{n-1} \neq c_{n-1}{}^2$, $d_{n-2} \neq c_{n-2}{}^2$, $d_{n-3} = c_{n-3}{}^2$, write down:

$$v^3 + c_{n-1}v^2 + c_{n-2}v + c_{n-3} = 0$$

Continue in this fashion until some equation is secured. Now, for the sake of example, suppose $d_{n-1} \neq c_{n-1}{}^2$, $d_{n-2} = c_{n-2}{}^2$, then we have:

$$v^2 + c_{n-1}v + c_{n-2} = 0$$

Now if $d_{n-3} = c_{n-3}{}^2$, then write:

$$c_{n-2}v + c_{n-3} = 0$$

If $d_{n-3} \neq c_{n-3}{}^2$, $d_{n-4} = c_{n-4}{}^2$, write:

$$c_{n-2}v^2 + c_{n-3}v + c_{n-4} = 0.$$

If $d_{n-3} \neq c_{n-3}{}^2$, $d_{n-4} \neq c_{n-4}{}^2$, $d_{n-5} = c_{n-5}{}^2$, write:

$$c_{n-2}v^3 + c_{n-3}v^2 + c_{n-4}v + c_{n-5} = 0.$$

The process is continued in the manner indicated. A set of equations is thus derived whose roots are known powers of those of (1.2). If the root squaring has been carried out far enough, the zeros of any one of the equations found by the above method have the same modulus.

As an example of the above process, suppose we have at some stage found the equation (8.3) and on root squaring of (8.3), come out with (8.4):

$$(8.3)\quad v^9 + 2v^8 - 3v^7 + v^6 - 4v^5 + 2v^4 - 5v^3 - 6v^2 + 4v + 25 = 0.$$

$$(8.4)\quad w^9 - 4w^8 + 2w^7 + 1w^6 + 16w^5 + 4w^4 + 1w^3 + 36w^2 + 1w + 625 = 0.$$

We examine the coefficients of the two equations and circle, as indicated, where the coefficients of (8.4) are squares of the coefficients of (8.3).

$$(8.3)\quad 1v^9 + 2v^8 - 3v^7 + 1v^6 - 4v^5 + 2v^4 - 5v^3 - 6v^2 - 4v + 25 = 0.$$

$$(8.4)\quad 1w^9 - 4w^8 + 2w^7 + 1w^6 + 16w^5 + 4w^4 + 1w^3 + 36w^2 + 1w + 625 = 0.$$

A B C D E F

Each of the circles is a starting point to secure the equations of Graeffe's method, and these are:

(8a) $v^3 + 2v^2 - 3v + 1 = 0$ (found by proceeding from circle A to circle B)

(8b) $v - 4 = 0$ (found by proceeding from circle B to circle C)

(8c) $-4v + 2 = 0$ (found by proceeding from circle C to circle D)

(8d) $2v^2 - 5v - 6 = 0$ (found by proceeding from circle D to circle E)

(8e) $-6v^2 - 4v + 25 = 0$ (found by proceeding from circle E to circle F).

If root squaring has been carried out far enough, the roots of 8a all have the same modulus, and likewise for the roots of 8b, of 8c, of 8d, and of 8e. Knowing that the roots of each equation have the same modulus leads to established techniques of solving as explained by Bodewig [9a], Brodetsky and Smeal [11], and Moore [32]. However, these methods are too varied and extensive to be described here and the reader is advised to consult further in the references.

A major source of annoyance is the recurring statement: "If Graeffe's method has been carried out far enough, ⋯". No simple criteria exist which determine the number of steps to use. Examples exist which show that certain coefficients reach the state of being a square of the corresponding preceding coefficients before others. Thus "settling" does not take place in the coefficients simultaneously and though Bodewig claims superiority for the method in that, for a_i real, it yields all the roots, real and complex, the question of how many times to root square still remains crucial in the practical application.

9. *Graphical Methods.*

(9a) [22]. Given (1.1), consider $y_1 = R_1(z)$, $y_2 = -R_2(z)$, such that $R_1(z) + R_2(z) \equiv P(z)$. If all the a_i are real, the real solutions of (1.1) are the x coordinates of the points of intersection of the graphs of y_1 and y_2.

(9b) [16]. Let: $z^2 - \xi z - \gamma$, be a quadratic factor of $P(z)$ and let z_1 and z_2 be the roots of: $z^2 - \xi z - \gamma = 0$. Hence, when z has either of these values, it follows that:

$$z^2 = \xi z + \gamma,\ z^3 = (\xi^2 + \gamma)z + \xi\gamma,\ z^4 = (\xi^3 + 2\xi\gamma)z + \gamma(\xi^2 + \gamma),$$

and in general:

$$z^n = \phi_{n-1} z + \gamma \phi_{n-2}, \quad \text{where:} \tag{9.1}$$

$$\phi_n = \sum \frac{(n-r)!}{r!(n-2r)!} \xi^{n-2r} \gamma^r. \tag{9.2}$$

Now in $P(z) = 0$, substitute for all expressions containing z to a degree higher than 1, the expression resulting by application of (9.1). This reduces $P(z) = 0$ to an expression of the form:

$$f_1(\xi, \gamma) z + f_2(\xi, \gamma) = 0, \tag{9.3}$$

which is valid for z_1 and z_2. Hence graph $f_1(\xi, \gamma) = 0$, $f_2(\xi, \gamma) = 0$ and determine their real intersections to determine a quadratic factor of $P(z)$.

(9c) [35]. Write $P(z) = F(z) + az + b = 0$. Let $F(z) = U + iV$, $u + iv = -az - b$. We find the roots of (1.1) as follows: Determine the intersections of

(*a*) Surface $U(xy)$ and plane $u(xy)$,

(*b*) Surface $V(xy)$ and plane $v(xy)$.

In general, (*a*) and (*b*) will be 2 curves. The intersections of these two curves give the roots of (1.1), where $z = x + iy$.

The inadequacies of curve sketching devices are usually apparent to the individual attempting to solve the problem: (9a) yields only real roots; plotting $f_1 = 0$, $f_2 = 0$ for (9b) may be extremely difficult due to the implicit representations of the functions; the graphing in (9c) may likewise be extremely difficult.

Another method, designed by Cornock and Hughes [13], approaches the solution through complex graphs and contours once an approximate answer is known.

10. *Concluding Remarks:*

A. An always converging iterative process is given in the following theorem of Rostovcew [40]:

Th. Let $F(x) = \sum_j A_j z^{m-j}$. Write $F(z) = G(z^2) + zH(z^2)$.

Suppose i) m is odd

ii) all $A_j > 0$, and, $\frac{A_1}{A_0} > \frac{A_2}{A_1} > \frac{A_3}{A_2} > \cdots > \frac{A_m}{A_{m-1}}$, for all A_j,

iii) $z_{k+1} = -\frac{G(z_k^2)}{H(z_k^2)}$,

then lim z_k exists and is the greatest or least root of $F(z) = 0$ according as $z_0 = 0$, or $z_0 = -A_1$.

One notes, of course, how limited the application of the theorem is, but it is a step in a desired direction.

B. Some of the many other available techniques, not widely used, are:

(10a) [16] Test function technique of Frazer and Duncan,

(10b) [27] Special devices for quartic and sextic equations,

(10c) [37] GCD method,

(10d) Inverse interpolation,

(10e) [47] Electrolytic tank method,

(10f) [41] Method of Lucas,

(10g) [14] Continued fraction approach of Frame,

(10h) [39] Convergent sequence method by function evaluation, of Rosenbloom,

(10i) [15] Method of Frank,

(10j) Method of Collatz (Using Hurwitz Reduction Theorem, unpublished (?) paper),

(10k) Horner's Method,

(10l) [60] Analog Method (using steepest descent).

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You said a plus number and a negative number cancel, didn't you?

n-GROUPS WITH IDENTITY ELEMENTS

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The generalized groups of W. Dörnte [1] are systems of elements with a polyadic operation satisfying an extension of the associativity and solvability axioms for ordinary abstract groups. This note is concerned with slightly stronger systems, which are defined by an extension of the associativity, identity, and inverse axioms for groups. Although the two concepts are not equivalent, it is considered worthwhile to indicate here part of the structure of these latter systems.

An n-semigroup is a system consisting of a nonempty set N and an $(n+1)$-composition in the set, satisfying the following axioms:

(1) For every $a_0, a_1, \cdots, a_n \epsilon N$, $a_0 a_1 \cdots a_n$ is an element of N. (Closure)

(2) $(a_0 a_1 \cdots a_n) a_{n+1} \cdots a_{2n} = a_0 \cdots a_{i-1} (a_i \cdots a_{i+n}) a_{i+n+1} \cdots a_{2n}$

for $i = 1, \cdots, n$. (Associativity)

An n-group is an n-semigroup satisfying in addition the axiom:

(3) For every $a, a_1, \cdots, a_n \epsilon N$, the equations

$$x a_1 \cdots a_n = a, \qquad a_1 \cdots a_n y = a$$

are solvable for x and y in N. (Solvability)

It is to be observed that a 1-group is an ordinary group. Since every group contains a special element called the identity, it is natural to investigate the generalization of this concept for $n > 1$. This was first done by E. L. Post [2], who showed that every n-group N contained a collection of n elements $e_1, \cdots, e_n$ satisfying the property that for all $a \epsilon N$, $a e_1 \cdots e_n = a = e_1 \cdots e_n a$. In this note, particular attention is drawn to the stronger systems in which all of these n special elements are equal to a single element. This special element is defined for our purposes in the following theorem.

Theorem. *Let an n-semigroup with a special element $e \epsilon N$ satisfy the following:*

(3.1) *For every $a \epsilon N$,* $\overbrace{e \cdots e}^{n} a = a$. (Left Identity)

(3.2) *For every $a \epsilon N$, there exists a $b \epsilon N$ such that*

$$b\overbrace{e \cdots e}^{n-1}a = e. \text{ (Left Inverse)}$$

Then this system is an n-group.

Proof. It is first shown that this system can be embedded in an ordinary group in such a way that the product of $n+1$ elements of N coincides with their product in the group.* For let G be the collection of all ordered pairs of the form (r, a), where r is a nonnegative integer and $a \epsilon N$. Define $(r, a) = (s, b)$ if and only if $r \equiv s$ (modulo n) and $a = b$, and the binary operation in G as follows:

$$(r, a)(s, b) = (r+s+1, \overbrace{e \cdots e}^{n-s-1}a\overbrace{e \cdots e}^{s}b),$$

where it is understood that $0 \leqq s < n$.

G is clearly a group under this binary operation with the (left) identity element $(n-1, e)$ and (left) inverse $(n-s-2, \overbrace{e \cdots e}^{s+1}b\overbrace{e \cdots e}^{n-s-1})$ of (s, a), where b is the left inverse of a with respect to e under (3.2).

Let G_0 be the complex of G consisting of the elements of the form $(0, a)$ for $a \epsilon N$. Then the system consisting of the set G_0 and the $(n+1)$-composition

$$(0, a_0)(0, a_1) \cdots (0, a_n) = (0, a_0 a_1 \cdots a_n)$$

is an n-group. For let $(r, x)(0, a_1) \cdots (0, a_n) = (0, a)$ in G. Since $(r, xa_1 \cdots a_n) = (0, a)$, it follows that $r \equiv 0 \pmod n$ and that $(r, x) \epsilon G_0$. Similarly for division on the right. Furthermore, this n-group is isomorphic with the given system. For let $\phi(0, a) = a$. ϕ is clearly a one-to-one mapping of G_0 onto N, preserving the operation. Hence, the given system is also an n-group, and the theorem is demonstrated.

The converse of this theorem is valid only if $n = 1$. Clearly both (3.1) and (3.2) are satisfied for ordinary groups. But for $n > 1$ the following example exhibits an n-group that does not contain a left identity element. Consider an ordinary cyclic group of order n^2 and primitive element a. Let $N = \{a, a^{n+1}, a^{2n+1}, \cdots, a^{(n-1)n+1}\}$. Define an $(n+1)$-composition in N by using the composition in the group taking $n+1$ elements at a time. This system is indeed an n-group. Except for the case $n = 1$, however, there is no special element satisfying (3.1). For $a^{n(kn+1)}a^{mn+1} = a^{(m+1)n+1} \neq a^{mn+1}$ for $n \neq 1$.

On the other hand, it is clear by axiom (3) that an n-group with a left identity element always satisfies (3.2). In addition, we also have the following.

Corollary. *Let N be an n-group with a left identity element e. Then,*
(3.1′) *For every* $a \epsilon N$, $a\overbrace{e \cdots e}^{n} = a$. (Right Identity)

(3.2′) *For every* $a \epsilon N$, *there exists a* $b' \epsilon N$ *such that* $a\overbrace{e \cdots e}^{n-1} b' = e$. (Right Inverse)

Furthermore, the elements b *and* b' *of* (3.2) *and* (3.2′), *respectively, are unique and equal.*

Proof. The first statement follows from the fact that for $a \epsilon N$, $(0, a) = (0, a)(n-1, e) = (0, a\overbrace{e \cdots e}^{n})$. The second is an immediate consequence of the fact that the system is an n-group. That b and b' are unique follows from the more general result that the solution of

$$a_0 a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n = a$$

exists uniquely (see also references [2] and [4]). For let (r, x) be the unique solution of

$$(0, a_0) \cdots (0, a_{i-1})(r, x)(0, a_{i+1}) \cdots (0, a_n) = (0, a)$$

in G. It is easily shown that $r \equiv 0 \pmod n$, and therefore $(r, x) \epsilon G_0$. Finally, $b = b\overbrace{e \cdots e}^{n} = b\overbrace{e \cdots e}^{n-1}(a\overbrace{e \cdots e}^{n-1} b') = (b\overbrace{e \cdots e}^{n-1} a)\overbrace{e \cdots e}^{n-1} b' = \overbrace{e \cdots e}^{n} b' = b'$.

Hence, the property that a left identity is a right identity in an ordinary group generalizes in this present study. Similar results are noted for the properties of inverses with respect to a given identity. The one important feature that does not generalize, however, is the uniqueness property of the identity. This is illustrated by the following example.

Let N be the collection of elements of an ordinary cyclic group of order np $(n > 1,\ p > 0)$ generated by a primitive element a. Define an $(n+1)$-composition by using the composition of the group, taking $n+1$ elements at a time. This system is an n-group, with n distinct elements $a^p, a^{2p}, \cdots, a^{np}$ all satisfying (3.1). For $a^r \cdots a^r a^s = a^{nr+s} = a^s$ if and only if $r = kp$ for some $k = 1, \cdots, n$. If in particular $p = 1$, then every element of N satisfies the identity property.

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Footnote

*Both Post [2] and Tvermoes [4] have shown that every n-group can be embedded in a group in such a way that the product of $n+1$ elements in the n-group coincides with their product in the group.

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A THEOREM CONCERNING THE BERNSTEIN POLYNOMIALS

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1. *Introduction.* This paper is concerned with some properties of the Bernstein polynomials, and it seems important to indicate just what these polynomials are before proceeding to the main results of the paper.

The famous approximation theorem of Weierstrass may be stated in the following fashion: If f is a real-valued and continuous function on an interval $[a, b]$, then for each $\epsilon > 0$ there exists a polynomial P such that $|f(x) - P(x)| < \epsilon$ for all x in the interval $[a, b]$.

This theorem was proved in a number of ways. Bernstein proved it by actually defining polynomials $B^f_k(x)$ of degree k associated with the function f over the interval $[0, 1]$. In fact, if f is bounded on $[0, 1]$ and continuous at a point x in this interval, then

$$\lim_{k\to\infty} B^f_k(x) = f(x).$$

The reader is referred to the interesting book *Bernstein Polynomials*, by G. G. Lorentz, Univ. of Toronto Press, 1953, for more information on these interesting polynomials.

The Bernstein Polynomials are defined by

$$B^f_k(x) = \sum_{i=0}^{k} \binom{k}{i} (1-x)^{k-i} x^i f\left(\frac{i}{k}\right). \tag{1}$$

2. *Theorem to be proved.* In making a study of the degree of approximation to f afforded by $B^f_k(x)$ as the index k is allowed to take larger values, the definition of $B^f_k(x)$ as given by relation (1) is sometimes rather unhandy because of the way in which k appears: as part of a binomial coefficient, an exponent on $(1-x)$ and as the denominator of a fraction in $f\left(\frac{i}{k}\right)$. It also is restricted to be integral and occurs as the upper index of summation. Consider now the simple case when f is a polynomial. The

Note: This paper was read before the December 1, 1956 meeting of the Md.-D.C.-Va. Section of the Mathematical Association of America. Sections 3 and 4 are based on the author's master's thesis written at the University of Virginia, June 1956.

theorem to be proved takes the following form:

Theorem. Let $f(x) = \sum_{n=0}^{\nu} a_n x^n$ be any polynomial in x of degree ν. Then there exist polynomials $Q^{\nu}_{\alpha}(x)$ of degree $\leqq \nu$ such that

$$(2) \qquad B^{f}_{k}(x) = f(x) + \sum_{\alpha=1}^{\nu} \frac{1}{k^{\alpha}} Q^{\nu}_{\alpha}(x),$$

and $Q^{\nu}_{\alpha}(x)$ does not depend on k. In fact $Q^{\nu}_{\alpha}(x)$ may be expressed in terms of Stirling numbers of the first and second kind.

The existence of $Q^{\nu}_{\alpha}(x)$ was shown by E. J. McShane, Annals of Mathematics Study No. 31, page 119, and it is proposed here to develop this in detail and actually exhibit the polynomials $Q^{\nu}_{\alpha}(x)$. However, it is necessary first to make a few remarks concerning the Stirling numbers.

3. *The Stirling numbers of the second kind.* For the purposes of this paper these are defined by the expansion

$$(3) \qquad x^n = \sum_{j=0}^{n} B^n_j \binom{x}{j}.$$

It is well known that

$$(4) \qquad B^n_j = (-1)^j \sum_{r=0}^{j} (-1)^r \binom{j}{r} r^n = \underset{x,\,1}{\Delta^j}\, x^n \Big]_{x=0},$$

and the reader is referred to *Vorlesungen über Differenzenrechnung,* by N. E. Nörlund, Berlin, 1924, reprinted by Chelsea, N. Y., 1954, or to *Calculus of Finite Differences,* by C. Jordan, N. Y., 1947, for a full discussion of the methods of finite differences used in showing these results concerning the Stirling numbers.

4. *The Stirling numbers of the first kind.* These are discussed second because they are most easily expressed in terms of the Stirling numbers of the second kind. For the purposes at hand they will be defined by the expansion

$$(5) \qquad \binom{x}{n} = \sum_{\alpha=0}^{n} C^n_{\alpha} x^{\alpha}.$$

Following Nörlund, these could be expressed in terms of Bernoulli polynomials of higher order:

$$C^n_{n-k} = \frac{1}{n!}\binom{n}{k} B_k^{(n+1)}(1), \tag{6}$$

$$\text{where}: \frac{x^n e^{tx}}{(e^x-1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(t)\frac{x^k}{k!}, \quad |x| < 2\pi.$$

Or, alternatively, one may proceed by using Cauchy's theorem to find:

$$C^n_\alpha = \frac{1}{2\pi i}\int_{\gamma:\,|z|=1}\binom{z}{n}\frac{1}{z^{\alpha+1}}\,dz, \tag{7}$$

$$= \frac{1}{2\pi i\alpha!}\int_\gamma \frac{z^\alpha e^z}{(e^z-1)^{n+1}}\,dz,$$

$$= \frac{\alpha}{n}\,\frac{1}{2\pi i\alpha!}\int_\gamma \frac{z^{\alpha-1}}{(e^z-1)^n}\,dz$$

Finally this leads to the well-known result that

$$n!\,C^n_{n-\alpha} = \binom{n-1}{\alpha} D_z^\alpha\left(\frac{z}{e^z-1}\right)^n\Bigg]_{z=0} \tag{8}$$

$$= (-1)^\alpha \sum_{j=0}^{\alpha}\binom{\alpha+n}{\alpha-j}\binom{\alpha-n}{\alpha+j}\frac{1}{j!}B_j^{j+\alpha}$$

and this last expression was first obtained by Ludwig Schläfli (Crelle's Journal für die reine und angewandte Math., Vol. 43, 1852, pp. 1-22; ibid. Vol. 67, 1867, pp. 179-182). It is also stated in Jordan's *Calculus of Finite Differences* on page 219.

Proceeding from an integral definition for C^n_α or for B^n_α, one may find upper bounds on the values of these coefficients, and an immense amount of material has appeared in the literature on bounds for the Stirling numbers, alternative definitions, etc.

This is all the information needed about the Stirling numbers except for the following easy evaluation:

$$B^n_n = n!, \qquad C^n_n = \frac{1}{n!}. \tag{9}$$

5. *Proof of the theorem in Section 2.* Consider the higher derivatives of $(e^y + z)^k$ relative to the variable y. By the binomial theorem,

$$(e^y + z)^k = \sum_{i=0}^{k} \binom{k}{i} e^{iy} z^{k-i},$$

$$D_y^n (e^y + z)^k = \sum_{i=0}^{k} \binom{k}{i} z^{k-i} i^n e^{iy}.$$

The right-hand member of this last expression may be put into another form, namely,

$$\sum_{j=0}^{n} \binom{k}{j} B_j^n \, e^{jy} (e^y + z)^{k-j}.$$

To see this, first note that one may as well sum on j from 0 to k since $\binom{k}{j}$ is zero for $j > k$ and B_j^n is zero for $j > n$. Then

$$\sum_{j=0}^{k} \binom{k}{j} B_j^n \, e^{jy} (e^y + z)^{h-j}$$

$$= \sum_{j=0}^{k} (-1)^j \sum_{i=0}^{j} (-1)^i \binom{j}{i} i^n \binom{k}{j} e^{jy} (e^y + z)^{k-j}$$

where relation (4) has been used.

$$= \sum_{i=0}^{k} (-1)^i i^n \sum_{j=i}^{k} (-1)^j \binom{j}{i}\binom{k}{j} e^{jy} (e^y + z)^{k-j}$$

$$= \sum_{i=0}^{k} (-1)^i i^n \binom{k}{i} \sum_{j=i}^{k} (-1)^j \binom{k-i}{j-i} e^{jy} (e^y + z)^{k-j}$$

$$= \sum_{i=0}^{k} i^n \binom{k}{i} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} e^{(j+i)y} (e^y + z)^{k-i-j}$$

$$= \sum_{i=0}^{k} \binom{k}{i} i^n e^{iy} \sum_{j=0}^{k-i} \binom{k-i}{j} (-e^y)^j (e^y + z)^{k-i-j}$$

$$= \sum_{i=0}^{k} \binom{k}{i} i^n e^{iy}(-e^y + e^y + z)^{k-i}$$

by the binomial theorem.

$$= \sum_{i=0}^{k} \binom{k}{i} i^n e^{iy} z^{k-i}$$

Therefore the following identity has been established:

$$\sum_{i=0}^{k} \binom{k}{i} z^{k-i} i^n e^{iy} = \sum_{j=0}^{n} \binom{k}{j} B_j^n e^{jy}(e^y + z)^{k-j}. \tag{10}$$

In this, take $z = 1 - x$, $y = \log x$ and one finds that

$$\sum_{i=0}^{k} \binom{k}{i} (1-x)^{k-i} x^i i^n = \sum_{j=0}^{n} \binom{k}{j} B_j^n x^j \tag{11}$$

Dividing each side by k^n one has further

$$\sum_{i=0}^{k} \binom{k}{i} (1-x)^{k-i} x^i \left(\frac{i}{k}\right)^n = \frac{1}{k^n} \sum_{j=0}^{n} \binom{k}{j} B_j^n x^j \tag{12}$$

The left-hand member is beginning to look like a Bernstein polynomial, thus far so good. However the right-hand side must be worked on to get the index k out of the binomial coefficient. Making use of relation (5) one finds

$$\frac{1}{k^n} \sum_{j=0}^{n} \binom{k}{j} B_j^n x^j = \frac{1}{k^n} \sum_{j=0}^{n} B_j^n x^j \sum_{\alpha=0}^{j} C_\alpha^j k^\alpha$$

$$= \frac{1}{k^n} \sum_{\alpha=0}^{n} k^\alpha \sum_{j=\alpha}^{n} B_j^n C_\alpha^j x^j$$

$$= \frac{1}{k^n} \sum_{\alpha=0}^{n} k^{n-\alpha} \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j$$

$$= \sum_{\alpha=0}^{n} \frac{1}{k^{\alpha}} \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j$$

Therefore one has the identity

$$\sum_{i=0}^{k} \binom{k}{i} (1-x)^{k-i} x^i \left(\frac{i}{k}\right)^n = \sum_{\alpha=0}^{n} \frac{1}{k^{\alpha}} \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j \tag{13}$$

And recalling the definition of $f(x)$ one has at once

$$B_k^f(x) = \sum_{i=0}^{k} \binom{k}{i} (1-x)^{k-i} x^i \sum_{n=0}^{\nu} A_n \left(\frac{i}{k}\right)^n$$

$$= \sum_{n=0}^{\nu} A_n \sum_{\alpha=0}^{n} \frac{1}{k^{\alpha}} \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j$$

$$= \sum_{\alpha=0}^{\nu} \frac{1}{k^{\alpha}} \sum_{n=\alpha}^{\nu} A_n \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j$$

$$= \sum_{n=0}^{\nu} A_n B_n^n C_n^n x^n + \sum_{\alpha=1}^{\nu} \frac{1}{k^{\alpha}} \sum_{n=\alpha}^{\nu} A_n \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j .$$

Recalling also that by relation (9), $B_n^n = n!$, $C_n^n = \frac{1}{n!}$, one finally obtains

$$B_k^f(x) = f(x) + \sum_{\alpha=1}^{\nu} \frac{1}{k^{\alpha}} Q_{\alpha}^{\nu}(x),$$

with

$$Q_{\alpha}^{\nu}(x) = \sum_{n=\alpha}^{\nu} A_n \sum_{j=n-\alpha}^{n} B_j^n C_{n-\alpha}^j x^j$$

which demonstrates the theorem and yields the explicit value of $Q_{\alpha}^{\nu}(x)$, and these polynomials may be studied to determine information about the degree of approximation given by the Bernstein polynomials.

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NUMBERS AND NUMBER SYSTEMS

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Every intelligent person has a background which includes experiences with numbers. These experiences go back to preschool days. A person learns to count before he learns to read. His concept of number grows from a concept which includes only the counting numbers to one which includes fractions, decimals, irrationals, and perhaps negative numbers and complex numbers. These are the numbers of elementary mathematics. These are the numbers of science and of business. These are the numbers of a world equipped with electronic computers, long-distance power-transmission lines, dial telephones, guided missiles, and space satellites.

Everyone takes numbers for granted. Everyone uses numbers as tools. Everyone has a number sense, an intuition about numbers and their relationships. But not everyone has spent time thinking about the nature of numbers. Our purpose in this paper is to focus primary attention upon numbers as objects of deliberate mature thought. We outline briefly the development of the number concept and the organization of numbers into number systems. We hope to stimulate the reader, to help him organize his past number experiences and perhaps help him to see something new and exciting about numbers.

A system of numbers consists of a set or collection of numbers and one or more operations for combining the numbers of the system. The numbers of the system sometimes are called the elements of the system. The system of counting numbers consists of the elements 1, 2, 3, ..., and the fundamental operations (addition, subtraction, multiplication, and division) for combining them. The symbol 1, 2, 3, ... suggests that there are an infinite number of elements; it also suggests that they have a natural order. This is the order in which they are used as counters. Following the natural number 1 is the natural number 2; 2 is sometimes called the successor of 1. Then 2 is followed by 3, 3 is followed by 4, and so on; 9 is followed by 10, 10 by 11, and so on; 1957 by 1958, and so on, ad infinitum. Although all of the natural numbers have never been written in symbols, they have been conceived, at least in a collective sense, as objects of thought.

Natural numbers developed as a sequence of distinguishable grunts or marks which the prehistoric man employed to see if his prehistoric dog had brought in all of his prehistoric sheep. These are the natural numbers

used in counting how many; these are the natural numbers used as cardinals. Natural numbers are also used as ordinals. For example, Mary ranked 3 in her graduating class. Enough about their use as ordinals and cardinals. Let us consider operations. As a short cut to counting the elements in a set which has been formed by the union of two other sets previously counted, the operation of addition developed. As a short cut for counting the elements of a set which results when one counted subset is removed from a counted set, there is the operation of subtraction. Multiplication may be considered as a short cut for repeated addition. Division may be considered as a short cut for repeated subtraction. These notions are familiar to everyone. But what properties does the natural number system have with regard to these fundamental operations? The mathematician lists some of them as follows. The system is closed with respect to addition. The system is closed with respect to multiplication. The system is not closed with respect to subtraction. The system is not closed with respect to division. In the system of natural numbers, addition is commutative; multiplication is commutative; addition is associative; multiplication is associative; and multiplication is distributive with respect to addition. In the sense of characterizing the system, that is, in the sense of furnishing a basis upon which to build the mathematical theory of the natural number system, this list of properties is incomplete. It is incomplete particularly in regard to the relationships which addition and multiplication have to subtraction and division. But this partial list suffices to give us a glimpse of the theory and to give us a basis for comparing this system with other systems.

THE NATURAL NUMBER SYSTEM

Closed with respect to addition.
Closed with respect to multiplication.
Addition is commutative: $a+b = b+a$
Addition is associative: $a+(b+c) = (a+b)+c$
Multiplication is commutative: $ab = ba$
Multiplication is associative: $a(bc) = (ab)c$
The distributive property: $a(b+c) = ab+ac$

What does it mean to say that the system is closed with respect to addition? This means that the result of adding 2 and 5 is a natural number, that the result of adding 1767253492233 to 1619175 is a natural number, indeed, that the sum of any two natural numbers is a natural number. Symbolically, if a and b are natural numbers, then $a+b$ is a natural number. In other words addition, as a binary operation which combines two numbers, is defined for every pair of natural numbers, and the result is a natural number. Incidentally, the mathematician considers $2+5$ a symbol

for a number. Indeed, it is a symbol for the same number which is frequently written as 7.

So the system is closed with respect to addition. Is it closed with respect to the other fundamental operations? No, it is not closed with respect to subtraction. It is impossible to subtract 7 from 2 in the system of natural numbers. The symbol $2-7$ is meaningless in the system of natural numbers. Yes, it is closed with respect to multiplication. The product of any pair of natural numbers is a natural number. If a and b are natural numbers, then the product of a and b, written as $a \times b$ or $a \cdot b$ or simply ab, is a natural number. No, it is not closed with respect to division. 7 cannot be divided by 5 in the system of natural numbers. Neither the symbol $7 \div 5$ nor the symbol $7/5$ has meaning in the system of natural numbers.

Let us look briefly at the implications of these closure properties as regards solving equations. In the system of natural numbers the equations $x-a=b$ and $\frac{x}{a}=b$ can always be solved for x; $x=a+b$ is the solution of the first one and $x=ab$ is the solution of the second one. Regardless of what natural numbers a and b denote, the symbols $a+b$ and ab also denote natural numbers. For the system is closed with respect to addition and multiplication. Thus the equations $x-17=7$, $x-2473=54617$, $x-246=1$, $\frac{x}{37}=5$, $\frac{x}{3}=53$, and $\frac{x}{13}=7540$, can all be solved in the system of natural numbers. But there are simple equations which cannot be solved in this system, for example, $x+2=1$, $x+95=6$, $2x+59=50$, $2x=1$, $3x=7$, and $17x=18+x$. The fact that these equations cannot be solved in the system of natural numbers emphasizes the property that the system is *not* closed with respect to subtraction and is *not* closed with respect to division.

Another important property of the system of natural numbers is that the sum and product of two natural numbers are the same regardless of the order in which they are combined. Thus $2+3=3+2$, $7\cdot 8=8\cdot 7$ and so on. In general, if a and b are natural numbers, then $a+b=b+a$ (this is the commutative property of addition) and $ab=ba$ (this is the commutative property of multiplication). Also listed are the associative properties of addition and multiplication. If a, b, c are natural numbers, then $a+(b+c)=(a+b)+c$ (this is the associative property of addition) and $a(bc)=(ab)c$ (this is the associative property of multiplication). At first addition and multiplication are conceived as binary operations, as operations which combine two numbers. Then $7+5+2$ means either $(7+5)+2$ or it means $7+(5+2)$. But $(7+5)+2=12+2=14$ and $7+(5+2)=7+7=14$, so that $(7+5)+2=7+(5+2)$. This is a particular instance of the associative property of addition. As an example of the associative property of multiplication,

note that $6(3\cdot2) = 6\cdot6 = 36$, $(6\cdot3)2 = 18\cdot2 = 36$, hence $6(3\cdot2) = (6\cdot3)2$. The property listed last is the distributive property: $a(b+c) = ab+ac$. For example, $2(3+5) = 2\cdot8 = 16$, and $2\cdot3+2\cdot5 = 6+10 = 16$, so that $2(3+5) = 2\cdot3+2\cdot5$.

As stated previously, these properties do not furnish all we need for developing the algebra of the natural numbers. But they are basic properties and upon them can be built a substantial portion of the theory of natural numbers. You might ask whether these properties can be proved. We have presented them today as a reflection of the experience which many people have had with numbers. We did not, however, mean to imply that these properties could not be proved. It boils down to the fact that a human being must start somewhere if he wishes to go somewhere. In this paper we start with the natural number system and we accept or postulate the properties listed above. The famous mathematician, Peano, started with a row of symbols in which each after the first had an immediate successor. He took the idea of successor as undefined and in terms of it he defined addition and multiplication. Using his definitions and axioms we could prove that $2+2 = 4$, for example. More generally, we could prove all the properties listed on chart 1. But even Peano started somewhere. Although we can prove that $2+2 = 4$ in Peano's theory, we cannot prove that $2+1 = 3$ in his theory. For $2+1$ is by definition the successor of 2 according to Peano; and of course the successor of 2 is the number 3 in the original list of symbols postulated by Peano.

In organizing numbers into systems it is convenient to consider next the system of integers: $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$. The three dots at each end of this symbol mean "and so on." In this natural order each integer has an immediate predecessor and an immediate successor; there is no beginning and there is no end. As eighth or ninth graders most of us met the integers for the first time. Perhaps we called them the signed whole numbers. We learned how to add, subtract, multiply, and divide signed numbers. Signed numbers have many uses in applications. If we wish to measure distances in two opposite directions from a starting point, we might denote distances in one direction as positive, distances in the other direction as negative. We might use positive and negative to distinguish between assets and liabilities, and between degrees above the freezing point of water and degrees below the freezing point of water. The arithmetic of integers is a powerful tool in solving many practical problems.

THE INTEGRAL NUMBER SYSTEM

$\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$

Closed with respect to addition.

Closed with respect to subtraction.
Closed with respect to multiplication.
Addition is commutative.
Addition is associative.
Multiplication is commutative.
Multiplication is associative.
Multiplication is distributive over addition.

As indicated in the above list the system of integers is closed with respect to addition, subtraction and multiplication, but not with respect to division. Indeed, $(-5)\div(-7)$ is not an integer. In the system of natural numbers we can solve the equation $x+5 = 8$, in which 5 and 8 are considered to be natural numbers. The answer is $x = 3$, the natural number 3. In the system of integers we can solve the equation $x+5 = 8$, in which 5 and 8 are considered to be integers, sometimes written as +5 and +8. The answer is $x = 3$, the integer 3, sometimes written as +3, In the system of natural numbers we cannot solve the equation $x+8 = 5$. In the system of integers we can solve the equation $x+8 = 5$. The answer is -3. Mathematically the system of integers has an advantage with respect to solving equations.

Note that the system of integers has the same commutative, associative, and distributive properties as those listed for the natural numbers. Can these properties be proved? Actually all of them can. We consider just one of them: addition is commutative in the system of integers, that is, if α and β are integers, then $\alpha+\beta = \beta+\alpha$. First we suggest a proof, organized into cases, using the definition of sum as taught in ninth-grade algebra.

Case 1. If α and β are positive integers, then $\alpha = +a$, $\beta = +b$, $\alpha+\beta = (+a)+(+b) = +(a+b)$, $\beta+\alpha = (+b)+(+a) = +(b+a)$ in which a and b are symbols for natural numbers. But $a+b = b+a$ in the system of natural numbers. Therefore, $+(a+b) = +(b+a)$ and $\alpha+\beta = \beta+\alpha$.

Case 2. If α and β are negative integers, then $\alpha = -a$, $\beta = -b$, $\alpha+\beta = (-a)+(-b) = -(a+b)$, $\beta+\alpha = (-b)+(-a) = -(b+a)$. But $a+b = b+a$ so that $\alpha+\beta = \beta+\alpha$.

Case 3. If α is positive, β is negative, $\alpha = +a$, $\beta = -b$, $a<b$, then $\alpha+\beta = (+a)+(-b) = -(b-a)$, $\beta+\alpha = (-b)+(+a) = -(b-a)$, so that $\alpha+\beta = \beta+c$.

Case 4. If $\alpha = +a$, $\beta = -b$, $a>b$, then $\alpha+\beta = (+a)+(-b) = +(a-b)$, $\beta+\alpha = (-b)+(+a) = +(a-b)$, so that $\alpha+\beta = \beta+\alpha$.

Case 5. If $\alpha = -a$, $\beta = +b$, $a<b$, then $\alpha+\beta = (-a)+(+b) = -(a-b)$, $\beta+\alpha = (+b)+(-a) = +(b-a)$, so that $\alpha+\beta = \beta+\alpha$.

Case 6. If $\alpha = -a$, $\beta = +b$, $a>b$, then $\alpha+\beta = (-a)+(+b) = -(a-b)$, $\beta+\alpha = (+b)+(-a) = -(a-b)$, so that $\alpha+\beta = \beta+\alpha$.

The remaining five cases, in which one or both of the numbers α, β is

0, may be proved in a similar manner. Since $\alpha+\beta = \beta+\alpha$ in all cases, it follows that addition is commutative in the system of integers. Similarly the other listed properties can be proved. But the ninth-grade rules for computing with signed numbers are inefficient for the purpose of developing the theory of integers. It is more expedient to define the integers as ordered pairs of natural numbers and to define addition and multiplication of integers using the ordered pair concept.

The idea behind this pair definition is that $5-3$ makes sense in the system of natural numbers; $3-5$ does not make sense in the system of natural numbers. So let us give it sense. But to avoid confusion between $5-3$ as an indicated subtraction in the system of natural numbers (that is, as another symbol for the natural number 2) and as a symbol for the integer 2 (or +2), we agree to think of (5, 3) (in which 5 and 3 are natural numbers) as one representation for the integer 2. We agree further that (6, 4), (10, 8), (17, 15), and (100, 98) are four more symbols denoting the same integer. (The two elements of each pair are, of course, natural numbers.) In fact, we define the integer 2 as the class of all ordered pairs of natural numbers (a, b) which have the property that $a-b = 2$, 2 a natural number. Similarly, we define the integer -2 as the class of all ordered pairs of natural numbers (a, b) which have the property that $b-a = 2$, 2 a natural number. Thus the class which defines the integer -2 includes the pairs (1, 3), (2, 4), (3, 5). Similarly we define the other positive and negative integers. We define 0 as the class of all ordered pairs of natural numbers (a, b) in which $a = b$.

Having defined the integers, we proceed to the operations. Thus we define the sum of two integers α, β as follows. Take any representations for α and β as ordered pairs of natural numbers: $\alpha = (a, b)$, $\beta = (c, d)$. Then $\alpha+\beta = (a+c,\ b+d)$ by definition.

Thus
$$(+5)+(-3) = (6, 1)+(1, 4) = (7, 5) = +2$$
$$(-5)+(+3) = (1, 6)+(4, 1) = (5, 7) = -2$$
$$(+5)+0 = (6, 1)+(1, 1) = (7, 2) = +5$$
$$0+(+5) = (1, 1)+(6, 1) = (7, 2) = +5$$

If a, b, c, d are any natural numbers, then $a+c$ and $b+d$ are natural numbers and hence $\alpha+\beta = (a+c,\ b+d)$ is a natural number. This shows that the system of integers is closed with respect to addition. Let us prove in terms of our new definitions of integer and sum of integers that addition is commutative in the system of integers.

Proof.
$$\alpha+\beta = (a, b)+(c, d) = (a+c,\ b+d)$$
$$\beta+\alpha = (c, d)+(a, b) = (c+a,\ d+b)$$

But $a+c = c+a$ and $b+d = d+b$ in the system of natural numbers. So $(a+c,$

$b+d) = (c+a, d+b)$ and $\alpha + \beta = \beta + \alpha$.

Notice the conciseness of this proof. Just one case, not eleven cases. When we define an integer as an ordered pair of natural numbers, and the sums and products of integers in terms of the sums and products of natural numbers, we are constructing the integers from the natural numbers. We are making the theory of the integers to depend upon the theory of the natural number. And this is efficient. One question does arise, however. This is in regard to the relationship of the integer 2, or +2, and the natural number 2. In practical applications the positive integers and the natural numbers seem to look alike. The integer 2 behaves like the natural number 2. Sums and products seem to be closely related. Thus $3+8 = 11$ and $3 \cdot 8 = 24$ are true in both systems, that is, they are true regardless of whether the 3, 8, 11, 24 are symbols for natural numbers or symbols for integers.

The positive integers have a relationship to the natural numbers which we describe in technical language by saying that the system of positive integers is *isomorphic* to the system of natural numbers. Briefly, if we mate each natural number n with the integer $+n$, then we may switch back and forth, from one system to the other, and still get correct sums and products. More precisely, if α and β are positive integers and their natural number mates are a and b in the scheme of mating just described, then the mate of $\alpha+\beta$ is $a+b$ and the mate of $\alpha\beta$ is ab.

We proceed now to the system of rational numbers. In college algebra a rational number is defined as a number which can be expressed as the quotient of two integers. Thus $\frac{7}{8}$ is a rational number. So are $\frac{5}{6}$, $\frac{-7}{3}$, 1.3 (equals $\frac{13}{10}$), 3.1416 (equals $\frac{31416}{10000}$), and $0.333\cdots$ (equals $\frac{1}{3}$). Actually the literal meaning of rational indicates that it has the property of a ratio, ratio-nal. Many persons think that rational numbers and fractions are the same. Actually they are not. According to the definition, every rational number can be written as a fraction in which the numerator and denominator are both integers. But not every fraction is a symbol for a rational number. Thus $\sqrt{2}$ is an irrational number; the fraction $\frac{\sqrt{2}}{1}$ is *not* a rational number. The word "fraction" does not denote a type of number; it denotes a type of symbol. Every number in elementary mathematics may be written as a fraction. For if "bloop" is a symbol for a number, then $\frac{\text{"bloop"}}{1}$ is another symbol for the same number.

Is an integer a rational number? Yes and no. The engineer says yes. $6 = \frac{6}{1} = \frac{12}{2}$. The mathematician says yes and no. In algebra we study the solution of polynomial equations with rational coefficients. The results

we obtain for these equations apply with equal force to equations with integral coefficient. For the purpose at hand, each integral coefficient might be considered as a rational number. On the other hand a systematic development of the properties of numbers is facilitated by differentiating between the integer 3 and the rational number 3. In the definition we are about to give, a rational number is constructed from integers and is logically distinct from an integer. Just as the natural number 2 and the integer 2 are distinct, so the integer 2 and the rational number 2 are also distinct.

Definition. A rational number is an ordered pair of integers $[a, b]$ with $b \neq 0$.

$$a^{\text{rational}} = [b^{\text{integral}}, c^{\text{integral}}] = [(d^{\text{natural}}, e^{\text{natural}}), (f^{\text{natural}}, g^{\text{natural}})]$$

$$2^{\text{rational}} = [2^{\text{integral}}, 1^{\text{integral}}] = [(3^{\text{natural}}, 1^{\text{natural}}), (2^{\text{natural}}, 1^{\text{natural}})]$$

Thus, if a and b are integers, $b \neq 0$. then $[a, b]$ is one representation for a rational number. Just as the fifth grader writes $\frac{10}{15} = \frac{14}{21} = \frac{2}{3}$, so we agree that $[10, 15]$, $[14, 21]$, $[2, 3]$ all denote the same rational number. We define equality by saying $[a, b] = [c, d]$ if $ad = bc$. We assume here that we have completed the theory of integers so that if a, b, c, d are symbols denoting integers, then ad and bc are integers and the statement $ad = bc$ has meaning from our knowledge of the system of integers. Thus, if we consider $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$ as symbols for integers, then $[-3, 2] = [-6, 4] = [9, -6]$, etc. Each of these four symbols denotes the same rational number which the engineer writes as -1.5.

THE RATIONAL NUMBER SYSTEM

Closed with respect to addition.
Closed with respect to subtraction.
Closed with respect to multiplication.
Closed with respect to division, except for division by 0.
Addition is commutative.
Addition is associative.
Multiplication is commutative.
Multiplication is associative.
Multiplication is distributive with respect to addition.

We proceed now to a consideration of some of the properties of the fundamental operations in the system of rational numbers. The sum of any two rational numbers is a rational number, the difference (in either order) of any two rational numbers is a rational number, the product of any two rational numbers is a rational number, and the quotient of any two rational

numbers is a rational number provided that the divisor in the quotient is not the rational number 0. Thus $\frac{2}{3}\cdot\frac{7}{8}=\frac{14}{24}=\frac{7}{12}$, $\frac{2}{3}\div\frac{7}{8}=\frac{16}{21}$, $\frac{2}{3}+\frac{7}{8}=\frac{16+21}{24}=\frac{37}{24}$, $\frac{2}{3}-\frac{7}{8}=\frac{16-21}{24}=-\frac{5}{24}$. Actually these closure properties and the other properties listed above are readily proved using the following definitions:

$$[a, b] + [c, d] = [ad+bc, bd],$$
$$[a, b] - [c, d] = [ad-bc, bd],$$
$$[a, b] \cdot [c, d] = [ac, bd],$$
$$[a, b] \div [c, d] = [ad, bc].$$

Since the system of integers is closed with respect to addition, subtraction, and multiplication, each of the symbols (appearing within the brackets in the right members of the above equations) $ad+bc$, $ad-bc$, ac, bd, is a symbol for an integer. Also, since b and d are each different from 0 (we assumed this when we stated that $[a, b]$ and $[c, d]$ were symbols for rational numbers), we know from our knowledge of integers that bd is different from 0. If in addition c is different from 0, then bc is different from 0, the symbol $[ad, bc]$ is a symbol for a rational number, and thus $[a, b] + [c, d]$ is a rational number. Thus the system of rational numbers is closed with respect to addition, subtraction, multiplication, and division, with one exception – division by 0 is not defined.

It is easy to prove that addition and multiplication in the system of rational numbers have the commutative, associative, and distributive properties. Suffice it here to prove that addition is commutative. If a, b, c, d are integers, $b \neq 0, d \neq 0$, then $\alpha = [a, b]$, $\beta = [c, d]$ are rational numbers and $\alpha+\beta = [a, b] + [c, d] = [ad+bc, bd]$, $\beta+\alpha = [c, d] + [a, b] = [cb+da, db]$. But in the system of integers, $ad+bc = cb+da$ and $bd = db$ so that $[cb+da, db] = [ad+bc, bd]$, that is, $\alpha+\beta = \beta+\alpha$, that is, addition is commutative in the system of rational numbers.

Let us go back to the question: are integers rational numbers? Having defined a rational number as an ordered pair of integers, the answer is no. But now we see, with more insight, perhaps, the relationship between the integers and certain rational numbers. Just as the computer and the engineer identify 4 and $\frac{4}{1}$ as being the same for purposes of computation, so we agree that the integer a and the rational number $\alpha = [a, 1]$ have a special relationship as far as computation is concerned. This relationship is made precise by saying that the system of all integers is isomorphic to the system of all rational numbers, each of which can be represented as $[a, 1]$ with a an integer.

As far as practical applications are concerned, rational numbers are essential tools in modern life. They are the numbers of measurement. The diameter of a piston is 3.574 inches. The weight of a liter of oxygen is

1.429 grams. Mathematically the rational numbers have an advantage over the integers as regards division. The existence of rational numbers makes possible the solution of equations such as $3x = 7$ and $5x + 8 = 0$. This raises the question: Why do we need more numbers? Suffice it to say that we do need other numbers for theoretical reasons, theoretical reasons whose by-products are many times practical – as practical as a rational number which is obtained as an approximate value of an irrational number by an engineer. The only "practical" way for the engineer to find this rational number may be to find first a number which is not rational. We need other numbers for theoretical reasons such as designing electrical circuits and networks so that we can enjoy using our dial telephones and watching our favorite TV programs. Mathematically we need to solve such equations as $x^2 = 7$, $x^2 = -7$, and $(x+2)^2 = -7$.

In the hierarchy of number systems we consider next the system of real numbers. For most practical purposes it is sufficient to consider a real number as any number which can be expressed as an infinite decimal. Most people have an intuitive feeling that infinite decimals are numbers as exemplified by such relations as $\frac{1}{3} = 0.33333\cdots$ or $\sqrt{2} = 1.41421\cdots$. But the very nature of the infinite decimal as a symbol may be somewhat baffling – particularly so if we attempt to develop the theory of real numbers in terms of numbers previously studied.

Two modern theories of real numbers are studied by practically every graduate student in mathematics. In the Cantor theory, a real number is defined as a class of sequences of rational numbers which have certain properties. Thus the real number 2 is a class of sequences of rational numbers. This class contains $(2, 2, 2, \cdots, 2, \cdots)$ as an element, $(1, 1.9, 1.99, 1.999, \cdots)$, as another element, and many other sequences as elements. Each sequence in this class may be considered as a rational number sequence representation of the real number 2. Having defined the real number we could next define the fundamental operations as applied to real numbers.

Thus if $\alpha = \{a_1, a_2, a_3, \cdots, a_n, \cdots\}$, $\beta = \{b_1, b_2, b_3, \cdots, b_n, \cdots\}$, then $\alpha + \beta = \{a_1 + b_1,\ a_2 + b_2,\ \cdots,\ a_n + b_n,\ \cdots\}$. It turns out that the system is closed with respect to the four fundamental operations except that division by 0 (the real 0) is not defined. It also turns out that the real numbers of Cantor enjoy all of the properties listed below.

THE REAL NUMBER SYSTEM

Closed with respect to addition.
Closed with respect to subtraction.
Closed with respect to multiplication.
Closed with respect to division, except for division by 0.
Addition is commutative.
Addition is associative.

Multiplication is commutative.
Multiplication is associative.
Multiplication is distributive with respect to addition.

In the system of real numbers the equation $x^2 = 2$ has a solution (in fact it has two roots). The system of real numbers furnishes us a number for the length of the diagonal of a square whose side is 1, namely $\sqrt{2}$.

A fundamental weakness of the system of rational numbers may be described in terms of internal and external criteria for the convergence of sequences. A sequence $a_1, a_2, \cdots, a_n, \cdots$, of numbers is convergent internally if, given any specified closeness,there is some number in the sequence beyond which any two numbers of the sequence differ by less than the specified closeness. Thus, if 0.01 is the specified closeness and if the sequence is the sequence of rational approximations to $\sqrt{2}$ (1, 1.4, 1.41, 1.414, ⋯) which we learned how to compute in the eighth grade, then 1.4 is a number of the sequence beyond which any two numbers of the sequence differ by less than 0.01. A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is convergent externally with limit a if, given any specified closeness, there is some number in the sequence beyond which any number of the sequence differs from the number a by less than the specified closeness. The sequence of rational approximations to $\sqrt{2}$, if considered as a sequence of real numbers, is convergent externally to $\sqrt{2}$. If the specified closeness is 0.001 then 1.41 is a number of the sequence beyond which any number of the sequence differs from $\sqrt{2}$ by less than 0.001. The weakness of the rational number system mentioned above is this: There are sequences of rational numbers which are convergent internally but which are not convergent externally with a rational limit, for example, the sequence of rational approximations to the positive square root of 2, the sequence we referred to just a moment ago. We might say that the real numbers are mathematically desirable, desirable in the sense that they provide limits for all internally convergent sequences of rational numbers.

We might ask then about sequences of real numbers which are convergent internally. Do we need to create some new numbers to provide limits for these sequences? No. A fundamental theorem in Cantor's theory is that any sequence of real numbers which is internally convergent is also externally convergent with a real number as its limit.

An equally important theory is the theory due to Dedekind. In a modern version of his theory a real number is defined as a lower segment of the rational numbers. For example, the real number 2 is defined as the class of all rational numbers which are less than the rational number 2; the real number $-\sqrt{2}$ is defined as the class of all negative rational numbers whose squares exceed 2. The fundamental operations can be defined in terms of these lower segments and the properties listed above can be proved. And although Dedekind and Cantor constructed real numbers from rational numbers in entirely different ways, the results are harmonious, harmonious in the sense that the resulting systems are isomorphic.

Finally, we need the complex number system. This is simple. A complex number may be defined as an ordered pair of real numbers. The number which the engineer writes as $3+\sqrt{2}\,i$ is written as an ordered pair of reals as $((3, \sqrt{2}))$. The number which the college student writes as i is written as an ordered pair in the form $((0, 1))$. We may define addition and multiplication in this system by writing $((a, b))+((c, d)) = ((a+c, b+d))$, $((a, b))\cdot((c, d)) = ((ac-bd, ad+bc))$.

THE COMPLEX NUMBER SYSTEM

Closed with respect to addition.
Closed with respect to multiplication.
Closed with respect to subtraction.
Closed with respect to division, except division by 0.
Addition is commutative.
Addition is associative.
Multiplication is commutative.
Multiplication is associative.
Multiplication is distributive with respect to addition.

Time permitting, we could define subtraction and division and develop the properties of this new system. It might be of interest to note that according to the definition of multiplication in this system, $((0, 1))\cdot((0, 1)) = ((-1, 0))$ and $((0, -1))\cdot((0, -1)) = ((-1, 0))$. Now $((-1, 0))$ is the complex number which behaves like the real number -1; the system of all complex numbers $((a, 0))$, with a and 0 real, is isomorphic to the system of real numbers. So the essence of the two equations just written is that -1 has two square roots in the system of complex numbers. These are the numbers which the college student writes as i and $-i$.

The system of complex numbers is a high point as a creation of the mind of man. The system is mathematically beautiful; mathematically it is complete in many desirable ways. Many theorems in the theory of these numbers and in the theory of functions of these numbers are statements of stark simplicity and beauty, devoid of special exceptions and a variety of cases. As an example, a polynomial equation of the nth degree with complex coefficients has exactly n roots (provided we follow a simple rule in regard to counting multiple roots). Thus $x^3-7=0$ has three roots, $x^5-7x^4+5x-15=0$ has five roots, $(2+i)x^{17}+(3+i)x^6+\sqrt{2}\,x-7=0$ has seventeen roots. The corresponding theory concerning the number of real roots of a polynomial equation with real coefficients is messy and complicated by comparison.

In summary, then, we have seen the numbers of elementary mathematics organized into systems. We started with the system of natural numbers. From them we created the integers. From the integers we created the rational numbers, from the rationals the reals, and from the reals the complexes. These are the numbers of elementary mathematics.

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THE LUCKY NUMBER THEOREM

D. Hawkins and W. E. Briggs

The lucky numbers of Ulam resemble prime numbers in their apparent distribution among the natural numbers and with respect to the kind of sieve that generates them [1]. It is therefore of interest to investigate their properties, bearing in mind the analogies to prime number theory.

The lucky numbers are defined by the following sieve. If S_n is an infinite sequence of natural numbers $t_{n,m}$ $(m = 1, 2, 3, \cdots)$, one obtains S_{n+1}, for $n > 1$, from S_n by removing every $t_{n,m}$ for which $t_{n,n}$ divides m. S_2 is the sequence $2, 3, 5, 7, 9, \cdots$ of the number 2 followed by the odd integers in increasing order of magnitude. S_1 is the sequence of natural numbers. The sequence of lucky numbers is $S = \lim_{n\to\infty} S_n$, that is, $2, 3, 7, 9, 13, 15, 21, \cdots$. This definition differs trivially from Ulam's.

Two properties are basic for the investigation of asymptotic properties of lucky numbers. By the definition if s_m represents the m-th lucky number, then

$$(1) \qquad s_m = t_{n,m} \qquad \text{for all } m < s_n .$$

This follows from the fact that $t_{n,s(n)}$ is the first number that will be removed from S_n in forming S_{n+1}, and is at the same time less than any number to be removed later on. Also by the definition, if $R(n, x)$ is the number of numbers not greater than x in S_n, and $[x]$ denotes the greatest integer in x, then

$$(2) \qquad R(n, x) = R(n-1, x) - \left[\frac{R(n-1, x)}{s_{n-1}}\right] n \geqq 2.$$

This fundamental recurrence relation has the following solution, in which $\{x\}$ denotes the fractional part of x and $\sigma_n = (1-1/2)(1-1/3)(1-1/7)\cdots(1-1/s_{n-1})$,

$$(3) \qquad R(n, x) = [x]\sigma_n + \sum_{i=2}^{n} \frac{\sigma_n}{\sigma_i}\left\{\frac{R(i-1, x)}{s_{i-1}}\right\}, \; n \geqq 2,$$

$$= [x]\sigma_n + E(n, x).$$

Clearly

(4) $$0 \leqq E(n, x) < n.$$

The properties of S now develop by a series of stages. The first is to find bounds for s_n. If one puts $R(n, s_{n+r}) = n+r$, which by (1) one may do for $0 \leqq r < s_n - n$, then

(5) $$s_n \leqq \frac{n}{\sigma_n}$$

by letting $r = 0$. From (5)

$$\frac{\sigma_{n+1}}{\sigma_n} = 1 - \frac{1}{s_n} \leqq 1 - \frac{\sigma_n}{n}, \quad n \geqq 2.$$

Putting $\rho_n = 1/\sigma_n$,

$$\rho_{n+1} - \rho_n \geqq \frac{\rho_{n+1}}{n\rho_n} > \frac{1}{n}, \quad n \geqq 2.$$

Summing from 2 to $n-1$, one obtains

$$\rho_n - \rho_2 > \sum_{t=2}^{n-1} \frac{1}{t}$$

which implies $\rho_n > \log n$, or

(6) $$\sigma_n < \frac{1}{\log n}, \quad n \geqq 2.$$

In $R(n, s_{n+r}) = n+r$ $(0 \leqq r < s_n - n)$, one may set $r = n$ and $r = n-1$, since $s_n > 2n$ for $n > 2$, which gives, by (3),

$$2n = s_{2n}\sigma_n + E(n, s_{2n})$$

$$2n-1 = s_{2n-1}\sigma_n + E(n, s_{2n-1}).$$

Hence, by (4) and (6), for $n > 2$,

(7) $$s_{2n} > n \log n$$

$$s_{2n-1} > (n-1)\log n.$$

It is now possible to show that the remainder term $E(n, x)$ is $o(n)$ when $x = s_n$. For from (1) and (3),

(8) $$R(n, s_n) = s_n \sigma_n + E(n, s_n) = n.$$

Let $\alpha(n)$ be the integer defined by $s_{\alpha(n)} \geqq n$ and $s_{\alpha(n)-1} < n$. By (7)

(9) $$\alpha(n) < \frac{3n}{\log n} \qquad n > n_0 .$$

Now split the sum $E(n, s_n)$ into two parts E_1 and E_2, where

(10) $$E_1 = \sum_{i=2}^{3n/\log n} \frac{\sigma_n}{\sigma_i}\left\{\frac{R(i-1, s_n)}{s_{i-1}}\right\}$$

(11) $$E_2 = \sum_{i=3n/\log n}^{n} \frac{\sigma_n}{\sigma_i}\left\{\frac{R(i-1, s_n)}{s_{i-1}}\right\}.$$

Clearly

(12) $$E_1 = 0\left(\frac{n}{\log n}\right).$$

In E_2, because of (1), put all $R(i-1, s_n) = R(\alpha(n), s_n) = n$, so that by (7)

(13) $$E_2 = 0\left(\sum_{i=3n/\log n}^{n} \frac{n}{i \log i}\right) = 0\left(\frac{n \log\log n}{\log n}\right).$$

It is now possible to write for $n \geqq 2$,

(14) $$\frac{\sigma_{n+1}}{\sigma_n} = 1 - \frac{1}{s_n} = 1 - \frac{\sigma n}{n + o(n)} .$$

Again using the substitution $\rho_n = 1/\sigma_n$, one obtains

$$\rho_{n+1} - \rho_n = \frac{\rho_{n+1}}{n\rho_n} + o\left(\frac{1}{n}\right)\frac{\rho_{n+1}}{\rho_n}.$$

By summation $\rho_n = \log n + o(\log n)$ and, therefore,

(15) $$\sigma_n \sim \frac{1}{\log n},$$

and from this, (1), and (3)

(16) $$s_n = \frac{n + o(n)}{\sigma_n} \sim n \log n .$$

(15) is the analogue of the Merten's theorem for prime numbers, and (16) is the analogue of the prime number theorem.

These results, especially (16), confirm a conjecture of one of the authors based on stochastic arguments [2]. They support the observation that the asymptotic distribution of prime numbers is not, except in details, a consequence of their primality, but characteristic of a wide class

of sieve-generated sequences, of which the lucky numbers are an example.

By using the results recursively in (14), S. Chowla has shown that the asymptotic value of (15) can be improved to

$$s_n = n \log n + \frac{n}{2}(\log \log n)^2 + o[n(\log \log n)^2]. \tag{17}$$

Since the corresponding result for prime numbers (where p_n is the n-th prime number) is

$$p_n = n \log n + n \log \log n + o(n \log \log n), \tag{18}$$

it follows, with only a finite number of exceptions, that $s_n > p_n$. With necessary calculations, this presumably will confirm Ulam's conjecture $s_n > p_n$ for all n.

REFERENCES

[1.] Verna Gardiner, R. Lazarus, M. Metropolis, and S. Ulam, *On Certain Sequences of Integers Defined by Sieves*, Math. Mag., vol. 29, (1956), pp. 117-122.

[2.] D. Hawkins, *Random Sieves*, Math. Mag. vol. 31, pp. 1-5.

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A SPECIAL CASE OF A PRIME NUMBER THEOREM

R. Lariviere

The prime number theorem that the exponent of the highest power of a prime p which divides $n!$ is equal to

$$N = [n/p] + [n/p^2] + [n/p^3] + \cdots,$$

where the series continues as long as the power of p is $\leq n$[1], has an unusually simple special case. This special case can be of interest to a calculus class that has found the powers of two troublesome in summing certain series or even to the general reader. It is stated as follows:

THE HIGHEST POWER OF TWO WHICH DIVIDES $2^m!$ IS

$$2^{2^m - 1}.$$

To prove this result we first drop the odd factors, since they cannot contribute, and have left $2 \cdot 4 \cdot 6 \cdot \ldots 2^m$. There are now $2^m/2$ factors, which may be written $(1 \cdot 2 \cdot 3 \cdot \ldots 2^{m-1})2^{2^{m-1}}$. Again dropping odd factors we obtain $(2 \cdot 4 \cdot 6 \cdot \ldots 2^{m-1})2^{2^{m-1}}$ where 2^{m-2} factors appear in the parenthesis. These may be written $(1 \cdot 2 \cdot 3 \cdot \ldots 2^{m-2})2^{2^{m-1} + 2^{m-2}}$. Repeating this process we finally obtain $(1)2^{2^{m-1} + 2^{m-2} + 2^{m-3} + \ldots + 2^{m-m}}$. The sum of the geometric series is $2^m - 1$. Hence the required power is

$$2^{2^m - 1}.$$

[1.] Trygve Nagell, *Introduction to Number Theory*, John Wiley, 1951, p. 47.

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MIKE WALLACE asks

GEORGE BERGMAN

What Makes a Genius Tick?

George Bergman is one of America's natural resources. This 14-year-old student at New York's Stuyvesant High School is already winning acclaim as an original mathematician. *Mathematics Magazine* has just published an original numbers theory evolved by George at the age of 12. The tall, gawky, talkative genius – his IQ is 205 – tosses abstract symbolisms around as vigorously as other kids swap baseball averages. To him, mathematics is one of the most exciting things in the world. But to the rest of us, George and others like him are exciting. The future of this nation lies in their hands.

Q. George, when did you first become interested in mathematics?
A. Let's see now ... oh, I have a memory of adding,doubling endlessly, when I was about 6.... 2 and 2 make 4 – and 4 and 4 make 8 – and 8 and 8 – you know, on for hundreds of steps.... Then when I was 7, I was playing around and discovered that if you double a number and take the square, it will be four times the square of the number. That's the first thing I ever wrote down.

Q. You have just published a paper in *Mathematics Magazine* which you wrote at the age of 12. What's it about?
A. It's called "A Number System With an Irrational Base."

Q. What's the point of it? Is it valuable somehow?
A. Oh, I don't think it has any application. It's a nice mental exercise.

Q. Can you explain what you like about mathematics, George? Most people don't have the vaguest understanding of it at all.
A. Well, mathematics is quite exciting. The excitement comes from ... well ...I don't know how to put it ... you know, when things fit together perfectly. When you prove something, when you've deduced something, and everything fits together ... well, it's just beautiful. It's terrific to discover patterns. It's all patterns really ... Of course, mostly, I get a great deal of excitement just telling someone about it.

Q. How many people can you possibly find to tell?
A. I found one person so far who can understand just about everything. He's a boy named Melvin Hochster. He also goes to Stuyvesant. He apparently knows as much mathematics as I do.

Q. How has your being a mathematician worked out in the American school system?
A. It hasn't had too much connection with my school work until recently. Arithmetic in the lower grades is a matter of acquiring skills. I wasn't so far ahead there. Arithmetical errors are still my weakest point.

Q. You mean a mathematician can be bad at arithmetic?
A. Easily. Mathematics is thoughts, ideas. But arithmetic! It's drudgery, just adding and multiplying and performing these boring operations over and over again. You make slips.

Q. Is this why so many people think they don't like mathematics?
A. Oh, yes. Also they don't like it because it's taught badly. Not enough connection is made between mathematics and the real world.

Q. What do you mean?
A. Well, after all, mathematics isn't a reality of its own. It's not a separate *something* in the universe. It's connected. It's a series of *ideas*. Most people never learn that. So they never find out how exciting a mathematical idea can be.

Q. Tell me, George, do you have a favorite mathematical idea–or is this a silly question?
A. Well, it's not like having a pet parakeet. But one thing I like very, very much is the fact that if you perform the same operation on conjugate numbers, the results will always be the conjugates. A more mathematical way of saying it is that there is an auto-morphism between conjugates.

Q. You're fond of this?
A. It's enchanting.

Q. George, when you wake up in the morning, what's the first thing you think about? Mathematics?
A. Oh, don't be silly. I think about breakfast.

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

I have found an apparent mistake in *Arithmetical Congruences with Practical Applications,* an article which appears in the March-April edition of the magazine, to which I would like to call your attention.

Under the listing of properties of congruences, one property reads: "d. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $a \equiv c \equiv b \equiv d \pmod m$."

Since the remainders, when a and c are divided by m need not by definition be the same, they are not necessarily congruent. This statement would be true if the additional condition of $a \equiv c \pmod m$ were added.

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Some Aspects of Multivariate Analysis. By S. N. Roy. John Wiley & Sons, 1958, 214 pages, $8\frac{1}{2} \times 11$ inch format. $8.00.

This is the latest addition to the Wiley Publications in Statistics, edited by Walter A. Shewhart and S. S. Wilks.

Although Professor Roy is concerned with "normal variate" data, he omits a discussion of point estimation. Similarly, while the testing of hypotheses is covered at length, the author places his emphasis on obtaining confidence bounds on certain parametric functions. The testing of hypotheses, in so far as it is developed, is largely a means to that end. The parametric functions that figure in Professor Roy's monograph are, in each case, a set of natural measures of departure from the customary null hypothesis, there being, in some simple situations, a single such function, and in some more complicated situations, a set of functions.

The first twelve chapters of the volume constitute a conscious attempt to lead up to confidence bounds on parametric functions, with detailed discussions following in the next two chapters. In the final chapter, Professor Roy extends multivariate analysis and analysis of variance to

categorical data. The extensive appendix covers a number of matrix theorems and various types of Jacobians and multiple integrals.

Professor of statistics at the University of North Carolina, the author is currently visiting professor at the University of Minnesota. He was previously with the Indian Statistical Institute and Calcutta University, and has been a visiting professor at several universities in the United States.

Richard Cook

Figurets : More Fun with Figures. By J. A. H. Hunter. Oxford University Press, 1958.

Here are 150 more fascinating mathematical puzzles which prove that figures can be fun. These problems for the adult layman are cast in the form of entertaining anecdotes. The author has included 16 Typical Solutions, illustrating the main mathematical concepts used, and a complete set of answers is provided.

'You want my age?' the lady cried,
Surprised and rather horrified.
'If that's the price I have to pay
To get a drink, why, then I'll say:
'When multiplied by just five more,
My age makes tum tum seven four.'
She smiled and shook her curly head.
'My grandson's nearly twelve years wed.'
The barman laughed. She got her drink.
But was she old enough, d'you think?

There are hours of entertainment in this book for the mathematically minded. It will delight all who meet Mr. Hunter's problems daily in the Toronto "Globe and Mail" or fortnightly in "Saturday Night" and is bound to create an even wider audience of Hunter fans. His feature column "Figuret" is now syndicated in the United States, Canada, the United Kingdom, India, and Australia.

Oxford University Press

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

341. *Proposed by James H. Means, Huston-Tillotson College.*

In the subtraction $STAR - RATS = TRSA$ the minuend, subtrahend and difference are composed of the same four digits. Replace the letters with numerical digits. Is the solution unique?

342. *Proposed by C.W. Trigg, Los Angeles City College.*

Show that $!n \equiv n! \pmod{n-1}$, where $!n$ is sub-factorial n.

343. *Proposed by Grant Heck, student at Lebanon Valley College, Pennsylvania.*

An old Hall and Knight problem reads, "If a is one of the 19 arithmetic means inserted between 2 and 3 and h is the corresponding harmonic mean, show that $a = 5 - 6/h$." Generalize this for any interval and for any number of inserted means and show that the correspondence is independent of the number of inserted means.

344. *Proposed by James McCawley Jr., Chicago, Illinois.*

Let $f(x)$ be a polynomial whose coefficients are integers. If the leading coefficient and constant term and an odd number of the remaining coefficients are odd, prove that the equation $f(x) = 0$ has no rational root.

345. *Proposed by D.A. Breault, Sylvania Electric Co., Waltham, Massachusetts.*

Given that $\sum_{n=1}^{\infty} \frac{1}{n^3} = s$ prove the following:

a) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = 3/4\, s$$

b) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = 7/8\, s$$

346. *Proposed by J.M. Gandhi, Jain Engineering College, Panchkoola, India.*

Find the value of the expression

$$\sqrt[3]{\left[11+4\sqrt[3]{\left\{14+10\sqrt[3]{\left(17+18\sqrt[3]{(\cdots)}\right)}\right\}}\right]}$$

347. *Proposed by Pedro A. Piza, San Juan, Puerto Rico.*

Let $\binom{n}{s}$ be the $(s+1)$st binomial coefficient of order n. Prove that

$$\binom{n}{1}1^n - \binom{n}{2}2^n + \binom{n}{3}3^n - \cdots (-1)^{n+1}\binom{n}{n}n^n = (-1)^{n+1}n!$$

SOLUTIONS

Consecutive Odd Integers

320. [November 1957] *Proposed by M.S. Krick, Albright College, Pennsylvania.*

Find three consecutive odd integers less than 10,000 which are divisible by 27, 25, and 7, respectively.

I. *Solution by Brother Edward Daniel, Xaverian High School, Brooklyn, New York.* If the integers are $2n-1$, $2n+1$ and $2n+3$ then we must have

(1) $$2n+3 \equiv 0 \bmod 7$$

(2) $$2n+1 \equiv 0 \bmod 25$$

(3) $$2n-1 \equiv 0 \bmod 27.$$

From (1) $n = 2+7g$ and (2) becomes

$$14g+5 \equiv 0 \bmod 25 \text{ and}$$
$$g = 5+25h.$$

Substituting in (3) we get

$$19-h \equiv 0 \bmod 27 \text{ and}$$

$$h = 19 + 27k.$$

Hence

$$2n - 1 = 6723 + 9450k$$

and the required integers are 6723, 6725 and 6727.

II. *Solution by Harry M. Gehman, University of Buffalo.* The conditions require the determination of three numbers: $xyz3$, $xyz5$, $xyz7$, where $z = 2$ or 7, and $xyz3$ is divisible by 27, and $xyz7$ is divisible by 7.

Suppose $z = 7$. Then necessary conditions are $x + y = 8$, (mod 9) and $10x + y = 3x + y = 0 \pmod 7$. The only solutions are (3, 5) and (9, 8). But 3573 and 9873 are not divisible by 27, and hence there is no solution when $z = 7$.

Suppose $z = 2$. Then necessary conditions are $x + y = 4$, (mod 9) and $100x + 10y + 2 = 0$, (mod 7) or $3x + y = 4 \pmod 7$. The only solutions are (0, 4) and (6, 7). We find that 423 is not divisible by 27.

Hence the unique solution is the set: 6723, 6725, 6727.

III. *Solution by Sam Kravitz, East Cleveland, Ohio.* Let $10a + 3$, $10a + 5$, and $10a + 7$ be the three integers. Then the conditions of the problem imply that: $\dfrac{10a+3}{27} = m$, $\dfrac{10a+5}{25} = m + k$, and $\dfrac{10a+7}{7} = m + k_1 + k_2$ where m, k_1, and k_2 are integers. From this we have

$$27m + 7(m+k_1+k_2) = 50(m+k_1)$$

or $16m = 7k_2 - 43k_1$. Now $27m - 3 = 25m + k_1 - 5$ or $2m = 25k_1 - 2$ so $16m = 200k_1 - 16$. Thus $k_2 = \dfrac{243k_1 - 16}{7}$ and $m = 25/2\ k_1 - 1$. Now when $k_1 = 20$ both k_2 and m are integers, $m = 249$ and $k_2 = 692$. From these we find that the three integers sought are 6723, 6725 and 6727.

Also solved by Felix A. Beiner, Western Electric Co., Cicero, Illinois; George Bergman, Stuyvesant High School, New York; D.A. Breault and M. Chaffin (Jointly) Sylvania Electric Co., Waltham, Massachusetts; Brother T. Brendan, St. Mary's College, California; J.W. Clawson, Collegeville, Pennsylvania; R.T. Coffman, Richland, Washington; Joseph D.E. Konhauser, Haller Raymond and Brown Inc., State College, Pennsylvania; Herbert R. Leifer, Pittsburgh, Pennsylvania; James H. Means, Huston-Tillotson College, Texas; Erich Michalup, Caracas, Venezuela; F.D. Parker, University of Alaska; Michael J. Pascual, Burbank, California; Charles F. Pinzka, University of Cincinnati; Lawrence A. Ringenberg, Eastern Illinois University; Barbara Steinberg, University of California at Berkeley; P.D. Thomas, Coast and Geodetic Survey, Washington, D.C.;

C.W. Trigg, Los Angeles City College; Dale Woods and J.T. Smith (Jointly) Idaho State College; Billy E. Yarbrough, Florence State College, Alabama and the proposer.

Area Construction

321. [November 1957] *Proposed by R.T. Coffman, Richland, Washington.*

Given a line of unit length and an acute angle θ, construct with compasses and straight-edge a plane figure having an area equal to

$$\int_0^\theta \cos^2\theta\, d\theta.$$

I. *Solution by Charles F. Pinzka, University of Cincinnati.* On the given line as diameter, construct a circle. With the given line as the initial side and one end as vertex, construct angles θ and $-\theta$ whose sides meet the circle. The sides of the angle of 2θ thus formed cut from the circle an area which is given in polar coordinates by

$$\tfrac{1}{2}\int_{-\theta}^{\theta} \cos^2\theta\, d\theta = \int_0^\theta \cos^2\theta\, d\theta \text{ as required.}$$

II. *Solution by Michael J. Pascual, Burbank, California.* Construct a circle with diameter of length $\sqrt{2}$ and lay off θ with its vertex on the circumference of the circle and a diameter as one of its sides. The section of the circle cut out by θ has the required area. To prove this, we set up polar coordinates with pole at the vertex of θ and polar axis along the diameter which forms one of its sides. The equation of the circle is $\rho = \sqrt{2}\cos\theta$ and the area cut out by θ is

$$\int_0^\theta \tfrac{1}{2}\rho^2\, d\theta = \int_0^\theta \cos^2\theta\, d\theta$$

III. *Solution by Joseph D.E. Konhauser, Haller, Raymond and Brown Inc., State College, Pennsylvania.* The integral is easily shown equal to $\theta/2 + 1/2 \sin\theta\cos\theta$. The compound figure consisting of a circular segment of radius 1 and angle θ added to a triangle having sides $1, \sin\theta$, $\cos\theta$ has area $\theta/2 + 1/2 \sin\theta\cos\theta$.

Also solved by D.A. Breault, Sylvania Electric Co., Waltham, Massachusetts; Brother T. Brendan, St. Mary's College, California; J.W. Clawson, Collegeville, Pennsylvania; Harry M. Gehman, University of Buffalo; M. Morduchow, Polytechnic Institute of Brooklyn; P.D. Thomas, Coast

and Geodetic Survey, Washington, D.C.; Lawrence A. R ngenberg, Eastern Illinois University; C.W. Trigg, Los Angeles City College; Dale Woods and Ray E. McLean (Jointly), Idaho State College and the proposer.

An Expression for e

322. [November 1957] *Proposed by D.A. Steinberg, Livermore, California.*

Show that if x is any real number, a any non-zero real number and p any non-negative integer, then

$$e^x = \left(\frac{a+x}{a}\right)^{p+1}\left[1 + \sum_{j=1}^{\infty} x^j \sum_{k=0}^{j} \binom{p+k}{k} \frac{(-1)^k}{a^k(j-k)!}\right].$$

Solution by Waleed A. Al-Salam, Duke University. We have

$$\left(1+\frac{x}{a}\right)^{-p-1} = \sum_{n=0}^{\infty} \binom{p+n}{n} \frac{(-1)^n x^n}{a^n}. \text{ Thus}$$

$$e^x\left(1+\frac{x}{a}\right)^{-p-1} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{n=0}^{\infty} \binom{p+n}{n} \frac{(-1)^n x^n}{a^n}$$

$$= \sum_{j=0}^{\infty} x^j \sum_{n+m=j} \binom{n+p}{n} \frac{(-1)^n}{a^n(j-n)!}$$

$$= 1 + \sum_{j=0}^{\infty} x^j \sum_{n=0}^{j} \binom{n+p}{n} \frac{(-1)^n}{a^n(j-n)!}$$

which is the required formula.

Also solved by Joseph D.E. Konhauser, Haller Raymond and Brown Inc., State College, Pennsylvania and Chih-yi Wang, University of Minnesota.

An Unstated Integer

323. [November 1957] *Proposed by Leo Moser, University of Alberta, Canada.*

Some problems which mention no integer explicitly have unique integer

solutions. For example, "What is the smallest perfect number?" has the solution, 6. Prove that for every positive integer n, there is a problem which mentions no integer explicitly, but which has n as its unique solu ion.

Solution by C.W. Trigg, Los Angeles City College. Consider the set A of problems such that every integer except n is a solution of at least one problem in the set. Then the problem "What integer does not appear among the solutions of the problems in set A?" has the unique solution, n. Thus n need not be restricted to positive values.

OR Consider the sequence B which contains all positive integers except n. Then the problem "What positive integer does not appear in sequence B?" has the unique solution, n.

OR Consider a problem C which has the unique solution, $-n$. Then the problem "Find the negative of the solution to problem C." has the unique solution, n.

OR Consider an array of positive integers in successive ordered groups each starting with a square number except the one which normally would contain n^2, where n^2 is placed at the end of the previous group. Then the problem, "What is the positive square root of the only perfect square that terminates a group?" has the unique solution, n. This same array could also yield a unique solution, k, to the question, "What is the number of the group which terminates in a square number?" (Clearly 1 would not be placed in an isolated group unless it were sought as the value for n.)

And so on *ad infinitum.*

Various solutions also submitted by Joseph D.E. Konhauser, Haller Raymond and Brown, Inc., State College, Pa.; Charles F. Pinzka, University of Cincinnati, Lawrence A. Ringenberg, Eastern Illinois University and the proposer.

The proposer mentioned that the problem is related to the Berry paradox and to the well known "proof" that every integer is very interesting.

Almost One

324. [November 1957] *Proposed by J.M. Howell, Los Angeles City College.*

Prove that:

$$\frac{\cos x}{1+\sin x} + \log(1+x) = 1 + 0(x^4), \quad 0 < x < 1.$$

Solution by M. Morduchow, Polytechnic Institute of Brooklyn. Let $f(x) = \cos x/(1+\sin x)$, $g(x) = \log(1+x)$ and $F(x) = f(x) + g(x)$. Then $f(0) = 1$

and $g(0)=0$. Moreover, by successively differentiating $f(x)$ and evaluating the derivatives at $x = 0$, it is found that $f'(0) = -1$, $f''(0) = 1$, $f'''(0) = -2$, $f''''(0) = 5$. Similarly, one finds $g'(0) = 1$, $g''(0) = -1$, $g'''(0) = 2$, $g''''(0) = -6$. Consequently, $F(0) = 1$, $F'(0) = F''(0) = F'''(0) = 0$ while $F^{iv}(0) = -1$. Hence, expanding $F(x)$ in a power (Taylor) series about $x = 0$, $F(x) = 1 - (1/4!)x^4 \cdots$.

Remarks. (a) By Lagrange's form of the remainder, the above analysis yields

$$R = |F(x)-1| = (|F''''(\xi)|/4!)x^4,$$

where for $0<x<1$, $0<\xi<1$.

(b) An alternative procedure would be to expand $\cos x$ and $\sin x$ in a power series and then obtain the power series for $f(x)$ by straight-forward division.

Also solved by Joseph D.E. Konhauser, Haller Raymond and Brown, Inc., State College, Pa.; P.D. Thomas, Coast and Geodetic Survey, Washington, D.C., and the proposer.

A Family of Ellipses

325. [November 1957] *Proposed by R.B. Kiltie, Montclair, N.J.*

Given the family of concentric ellipses

$$x^2/A^2 + A^2y^2/M^2 = 1.$$

(1) Show that each has the area πM.

(2) Find the points at which the ellipses are tangent to the equilateral hyperbolas $4x^2y^2 = M^2$.

(3) Show that through every point for which $4x^2y^2 < M^2$ there pass two ellipses.

Solution by Michael J. Pascual, Burbank, California. Solution:

(1) Using the parametric representation $y = \dfrac{M}{A}\sin\theta$, $x = A\cos\theta$ for the ellipses and the line integral representation for the area we get $A = \int_C \dfrac{x\,dy - y\,dx}{2}$ or

$$A = \tfrac{1}{2}\int_0^{2\pi} (M\cos^2\theta\,d\theta + M\sin^2\theta\,d\theta) = \tfrac{1}{2}\int_0^{2\pi} M\,d\theta = \pi M.$$

2) By differentiating the equations for each family and eliminating y' we

obtain $x = \pm\dfrac{A}{\sqrt{2}}$, $y = \pm\dfrac{M}{4x}$. Hence the four points of tangency must be

$$\left(\frac{A\sqrt{2}}{2}, \pm\frac{M\sqrt{2}}{4A}\right) \text{ and } \left(\pm\frac{A\sqrt{2}}{2}, \frac{M\sqrt{2}}{4A}\right)$$

(3) By differentiating with respect to x the equation of the ellipses and then eliminating A between the two equations we get

$$x^3y(y')^2 + (M^2 - 2x^2y^2)y' + xy^3 = 0$$

so if $(M^2 - 2x^2y^2)^2 - 4(x^3y)(xy^3) > 0$, then there are two real distinct values for y' so that two ellipses pass through each such point. Simplifying this relation we get $M^2 > 4x^2y^2$.

Also solved by Joseph D.E. Konhauser, Haller Raymond and Brown, Inc., State College, Pa.; Dale Woods, Idaho State College (partially) and the proposer.

An Alternating Harmonic Series

326. [November 1957] *Proposed by Erich Michalup, Caracas, Venezuela.*

Sum the series

$$1 - 1/9 + 1/17 - 1/25 + 1/33 - \cdots$$

I. *Solution by Chih-yi Wang, University of Minnesota.* Let the partial sum of the first N terms be denoted by $S(N)$. Then

$$S(N) = \int_0^1 \frac{1-(-t^8)N}{1+t^8}\, dt$$

As $N \to \infty$, $S(N) \to S$, where

$$S = \int_0^1 \frac{dt}{1+t^8}$$

Let $\propto = \sqrt{2-\sqrt{2}}$ and $\beta = \sqrt{2+\sqrt{2}}$. By resolving $(1+t^8)^{-1}$ into partial fractions:

$$\frac{1}{1+t^8} = \frac{At+B}{t^2+\propto t+1} + \frac{Ct+D}{t^2-\propto t-1} + \frac{Et+F}{t^2+\beta t+1} + \frac{Gt+H}{t^2-\beta t+1}$$

we get

$A = -C = (\sqrt{2}-1)/4\sqrt{2}\propto$, $B = D = 1/4$; $E = -G = (\sqrt{2}+1)/4\sqrt{2}\beta$, $F = H = 1/4$.

Hence

$$S=\frac{\sqrt{2}-1}{8\sqrt{2}\,\alpha}\ln\frac{2+\alpha}{2-\alpha}+\frac{\sqrt{2}+1}{8\sqrt{2}\,\beta}\ln\frac{2+\beta}{2-\beta}+\frac{\pi}{8\sqrt{2}}\left(\frac{\sqrt{2}+1}{\alpha}+\frac{\sqrt{2}-1}{\beta}\right).$$

II. *Solution by Edgar Karst, IBM Scientific Computing Laboratory, Endicott, New York.* For the purpose of using numerical computation methods the above series may be written: $1 - 1/8(1-1/2+1/3-1/4+1/5-1/6+\cdots) + 0.0111111111\cdots = 1 - 1/8 \log_e 2 + 1/90 = 1 + 1/90 - 0.6931471806/8 = 1.0111111111\cdots - 0.0866433976\cdots = 0.9244677135\cdots$.

Also solved by J.W. Clawson, Collegeville, Pa.; Joseph D.E. Konhauser, Haller, Raymond and Brown, Inc. and the proposer.

Konhauser pointed out that the series is a special case of the series $\frac{1}{a}-\frac{1}{a+d}+\frac{1}{a+2d}-\cdots$ and applied the formula for the sum of the generalized harmonic series

$$S=\frac{\pi\csc\frac{\pi a}{2}}{2d}-\frac{2}{d}\sum_{i=1}^{d/2}\frac{\cos(2i-1)\pi a}{d}\cdot\ln\sin\frac{(2i-1)\pi}{2d}$$

Comment on Problem 283

283. [September 1956 and January 1958] *Proposed by Jack Winter and Richard Kao, The Rand Corporation, Santa Monica, California.*

Comment by R.M. Conkling, New Mexico A and M College.

In Vol. 31, No. 3, Jan-Feb., 1958, Paul Pepper generalizes Problem 283 of Vol. 30, No. 1, to arrive at the identity

$$\sum_{a_n=0}^{N}\sum_{a_{n-1}=0}^{a_n}\cdots\sum_{a_1=0}^{a_2}1=\binom{N+n}{n}$$

Whenever one sees a binomial coefficient, can a combinatorial problem be far removed? Consider the sum

$$\sum_{a_n=0}^{N}\sum_{a_{n-1}=0}^{a_n}\cdots\sum_{a_1=0}^{a_2}k_{a_1a_2a_3\cdots a_n}$$

The terms in this sum have indices satisfying $0\leqq a_1\leqq a_2\leqq\cdots\leqq a_{n-1}\leqq a_n\leqq N$, as Pepper notes. Thus, there are as many sets of indices as there are monotone non-decreasing sequences formable from the integers $0,1,2,\cdots,N$.

This is the same as the number of combination of $N+1$ things taken n at a time, allowing repetitions. There are $\binom{N+1+n-1}{n}$ of these. Take $k = 1$ for all sets of indices, and Pepper's result follows.

The original problem with answer $\binom{2n}{n}$, for example, is equivalent to counting the number of monotone non-decreasing sequences formable from $0, 1, 2, \cdots, n$, which, in turn, is equivalent to the problem of counting the number of combinations of $2n$ objects taken n at a time, as the answer indicates. If the $2n$ objects are ordered, each monotone sequence has for its first member the ordinal of the first object used in the combination. Its subsequent members are one less than the differences of the ordinals of the objects used.

Quickies

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 224. Find the sum to N terms of $1, 12, 123, 1234, \cdots a_n, a_n + 1111\cdots11$ [*Submitted by M.S. Klamkin.*]

Q 225. If m and n are positive integers, then if either 3^m+1 or $3^{m+4n}+1$ is a multiple of 10, then the other is also. [*Submitted by C.W. Trigg.*]

A 225. $3^4 = 81$ ends in 1, so 3^{4n} ends in 1 also. Hence, if 3^m ends in 9, so must $(3^m)(3^{4n})$, and conversely. It follows that m is of the form $4k+2$.

$$S = \sum_{r=1}^{N} \sum_{s=1}^{r} \frac{10^s-1}{9} = \sum_{r=1}^{N} \frac{10(10^r-1)}{9^2} - \frac{r}{9} = \frac{10^2(10^N-1)}{9^3} - \frac{10N}{9^2} - \frac{N(N+1)}{18}$$

Thus

1, 10, 100, 1000, $\cdots$

1, 11, 111, 1111, $\cdots$

A 224. The successive differences are

Answers

Trickies

A trickie is a problem whose solution depends upon the perception of the key **word,** phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 22. How can a cube be turned on an ordinary lathe? [*Submitted by*

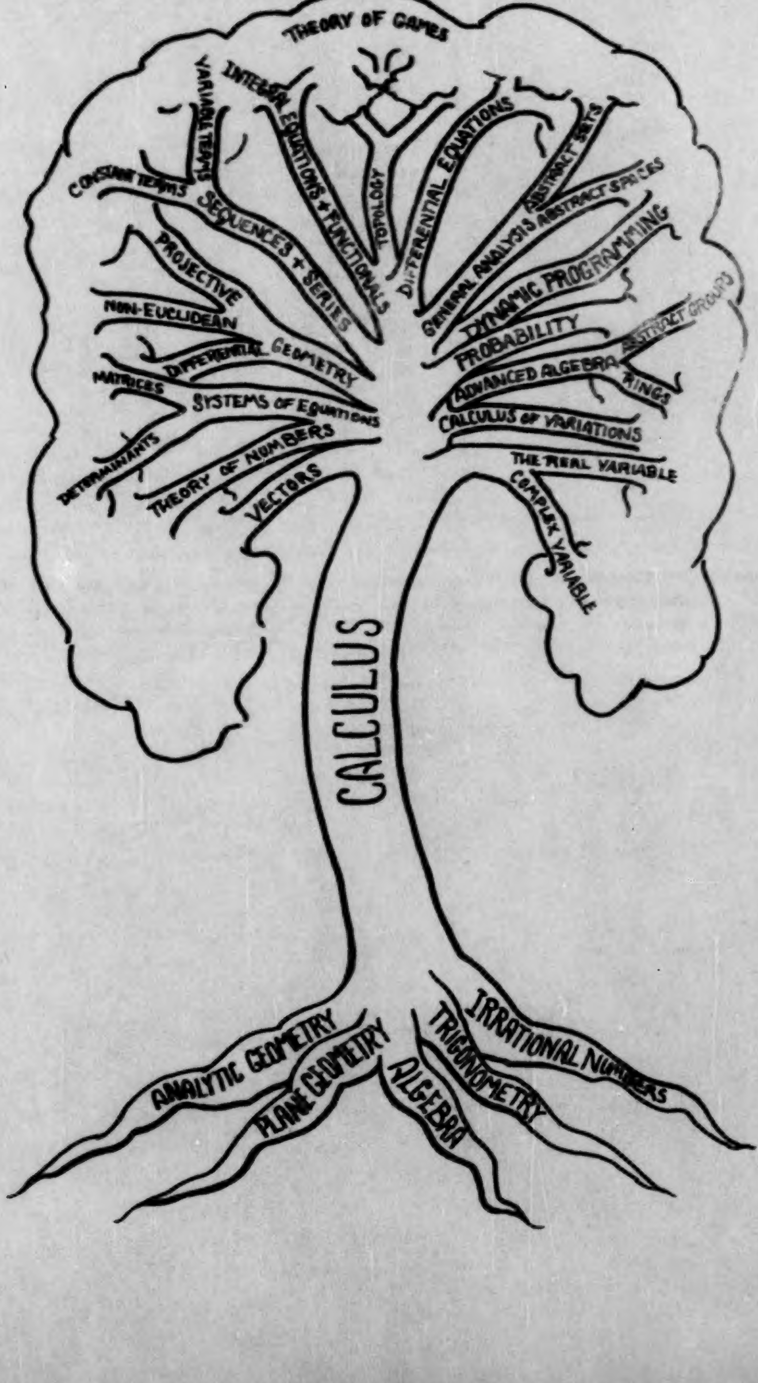
Richard K. Guy.]

T 33. One amoeba divides every 30 seconds. Under these conditions it takes 24 hours to fill a test tube. If we started with two amoebae, how long would it take to fill the same test tube? [*Submitted by Stanley L. Ksanznak.*]

Solutions

S 32. Start with the cutting tool perpendicular to the axis of rotation. A plane face is produced. Repeat with the material suitably oriented to produce the other six faces.

S 33. Starting with two amoebae is equivalent to starting at the end of the first 30 seconds, hence 23 hours, 59 minutes and 30 seconds would be required.



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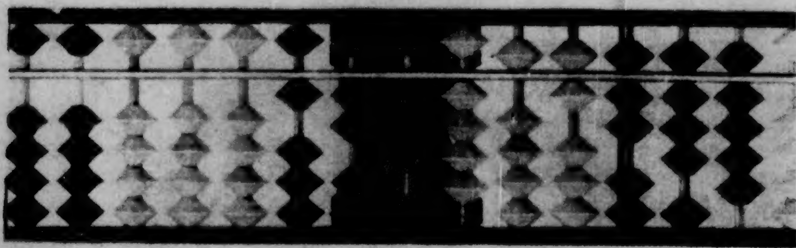

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